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Factors Affecting Reattachment of Supersonic Flows

by

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NOTATION

List of Symbols

$\gamma = c_p/c_v$	ratio of the specific heats of the fluid under consideration
σ	mixing parameter (see page 12)
$\mathcal{X}, \mathcal{Y}$	rectangular coordinate system for points in the physical plane, orientated so that the $\mathcal{X}$-axis coincides with the theoretical boundary of the jet exhausting into a perfect fluid
x, y	local coordinate system linked directly to the local velocity profile
ℓ, m	curvilinear coordinate system for points in the physical plane based on the families of Mach lines
δ	physical thickness of the boundary layer
D	diameter of the afterbody of a configuration
h	height of separated-flow region, or radius of the blunt-ended body
r	radius of curvature of the leading edge
η_p	position parameter (see page 9)
$\zeta = y/\delta$	reduced normal coordinate
$\eta = \zeta \cdot \eta_p$	reduced position coordinate
λ, μ	epicycloidal coordinates applicable in the hodograph plane
ψ	angle that the velocity vector makes with a reference direction selected in the physical plane
Ψ	reattachment angle (see Figure 1)
ϵ	angle of attack of a flat plate (see page 61)
α	Mach angle ($\sin \alpha = 1/M$).
X	curvature of a streamline
u	local velocity
U	hypothetical velocity (see Eq. 4' on page 14)

φ	reduced velocity, $\varphi = u/u_1$
K	reduced velocity occurring on the discriminating streamline
a	speed of sound
w_1	limiting velocity of the flow, $w_1 = \sqrt{\frac{2}{\gamma - 1}}\, a_i$, when expanded to a vacuum
M	Mach number
P(M)	Prandtl-Meyer pressure function (see page 21)
Σ (M)	areal variation, as a function of Mach number, for a streamtube subjected to isentropic flow processes
p	pressure
ρ	density
T	temperature
$\Theta = T_1/T$	temperature profile described on page 3
H	enthalpy
S	entropy
s	reduced entropy, $s = S/\gamma R$
s_n	entropy gradient taken in a direction lying normal to a streamline
G	tangential gradient of the pressure along a streamline (see page 65)
q	mass flow
C_q	mass flow coefficient, $C_q = q/(\rho_1 u_1 h)$
J	momentum exchange
C_μ	momentum exchange coefficient, $C_\mu = 2J/(\rho_1 u_1^2 h)$
Λ	coefficient signaling the influence of streamline curvature (Λ_p indicates the curvature coefficient in plane two-dimensional flow, Λ_r indicates the curvature coefficient for axially symmetric flow)
δ	actual thickness of the momentum boundary layer
δ_1	displacement thickness of the boundary layer
δ_{1_i}	displacement thickness of the boundary layer in incompressible flow (see page 47)
δ_2	momentum thickness of the boundary layer

Indices

∞	refers to the undisturbed flow far upstream from any obstacle
0	refers to the state of the flow which is attained just ahead of the blunt-ended base of the body about which an expansion is to take place
1	refers to the inviscid and constant pressure region of the flow which is presumed to exist outside of the mixing zone downstream of the expansion corner
2	refers to the state of the flow attained downstream of the reattachment point
i	relates to conditions obtained by bringing the flow to a stop isentropically
j	indicates a jet boundary
ℓ	indicates the discriminating streamline for the reattachment flow
N	denotes a streamline serving as a reference boundary
C	denotes any arbitrary general streamline in the flow
c	refers to pressure existing on the base of the body; i.e., p_c
—	a barred symbol signifies that it pertains to the reference ideal configuration; i.e., the boundary layer is assumed to be practically non-existent

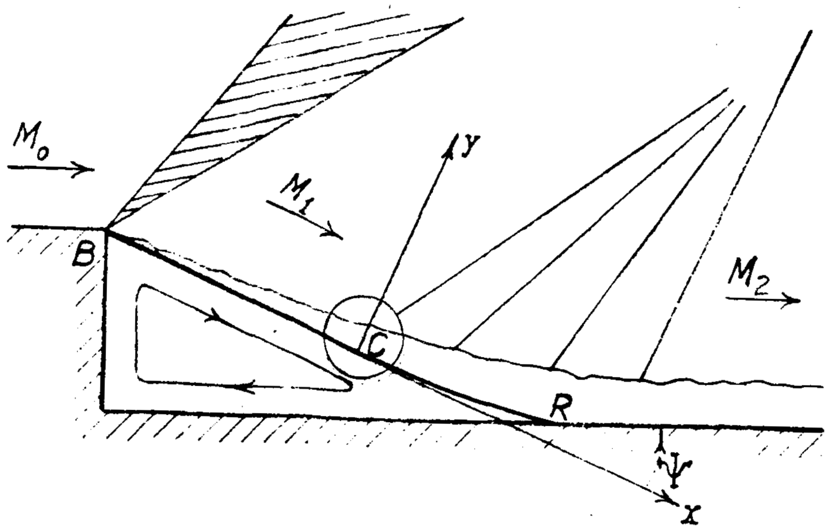
SUMMARY

The theoretical results are first reviewed that pertain to prediction of base pressures, as obtained by several authors [1,8]. A linearized method is then presented for calculation of the influence exerted by the several contributing factors upon the conditions found at reattachment. In particular it has been found possible to deduce a very simple relation linking the effect of a boundary layer upon the reattachment with the influence of a jet injected into the dead-water region.

A calculational procedure is then recommended for determination of the influence that certain parameters have in shaping the curved jet boundary. This method is especially suitable to the study of reattachment in cases where a circular jet of not just the ordinary type is to be treated (i.e., where entropy non-uniformities are to be accounted for, or where the rear end of the body is curved appreciably).

I. INTRODUCTION

In Figure 1 a schematic model of supersonic flow experiencing reattachment is illustrated, for the case of a plane two-dimensional configuration in order to make the concepts most precise. This way of representing and discussing the phenomenon associated with reattachment of a supersonic stream seems most direct and unequivocal for present purposes. The stream separates from the rearward facing step at point B and reattaches itself to the wall at point R. In bridging this hollow, therefore, the energetic flow has imprisoned a dead-water region composed of very feeble recirculating currents, having a relatively low speed, in comparison to the average unretarded speed representative of the passing energetic flow.



Flow Conditions at Point C

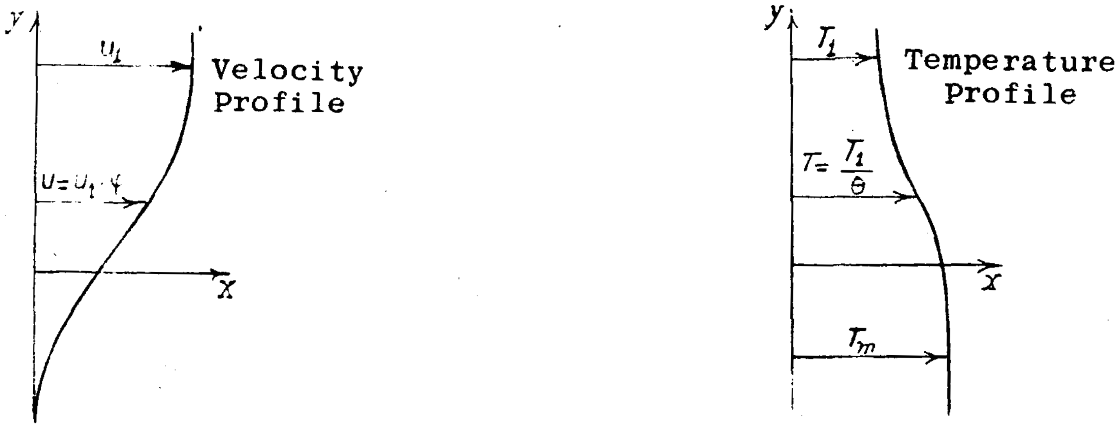


Figure 1

Flow Past a Rearward-Facing Step in Two-Dimensional (Plane) Case

Experimental research on these matters demonstrates that along the path B to C the flow next to the dead-water region experiences practically no pressure gradient (the flow is isobaric) while the sequential laminae or fluid layers lying near to the boundary between the separated flow and the dead-water region resemble a jet in which the speed varies rapidly and continuously as one makes a traverse in a direction normal to the jet flow lines. This region of rapid change in stream velocity is called the "mixing zone" because under the influence of viscosity various transport processes take place between the external energetic stream and the inner dead-water flow. In this manner not only is momentum transferred from one layer to the next but even conduction and diffusion phenomena are brought into play.

In consequence, the character of the flow at the time that the reattachment starts to take place at point C may be designated by two quantities: one is the mean direction of the approaching flow which makes the angle Ψ with the final direction assumed after reattachment, and the second is the local distribution of velocities and densities (or temperatures) in the mixing region, which are represented symbolically by the relations

$$\frac{u}{u_1} = \varphi(y) \text{ and } \frac{\rho}{\rho_1} = \frac{T_1}{T} = \theta(y).$$

On closer examination of the flow field it will be observed that there is one particular streamline (ℓ) which leads to the stagnation point R. This discriminating streamline is so situated that all streamlines which lie inboard of (ℓ), i.e., for $y < y_\ell$, will, of necessity, be turned back into the dead-water zone as the flow approaches the point R, while all the streamlines which lie outboard of (ℓ), i.e., for $y > y_\ell$, will find their way on downstream past the point R.

If no suction or blowing is permitted to disturb the streamline pattern in the dead-water region it is quite clear that the discriminating streamline (ℓ) will coincide with the jet streamline (j) which issues from the corner B. If, however, a certain steady stream of fluid is injected into the dead-water zone, then a steady exchange of flow will take place so that an equal quantity of fluid to that introduced upstream will escape downstream at the location R. In this instance the discriminating streamline (ℓ) will be situated inboard of the jet boundary j. In the opposite case, when suction is applied in the dead-water zone, the amount of fluid which is withdrawn from the dead-water region has to be compensated for by skimming off an equivalent amount of fluid from the outboard flow. Thus, in this instance the discriminating streamline must lie farther outboard than the jet boundary, j. See Figure 2.

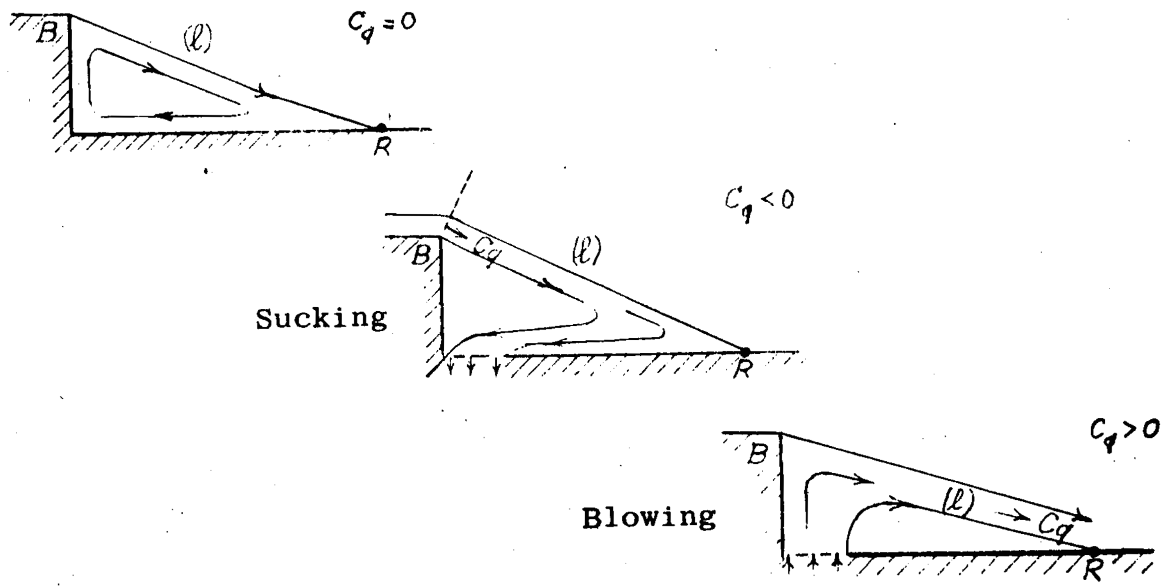


Figure 2

Discriminating Streamlines for Base Flows with Different Types of Ventilation

A number of authors have made use of the assumption, called the "capture" hypothesis, which premises that the stagnation pressure (which results if the flow is brought to rest isentropically), associated with the discriminating streamline (ℓ) as a result of the mixing processes which are taking place in the separated flow, must be set equal to the pressure p_2 which is attained by the external flow after it has undergone a deflection through the angle Ψ. This hypothesis, although strictly true only for a rather idealized and simplified version of the reattaching flow, nevertheless, has the advantage of establishing a working relation between the flow parameters Ψ, Θ_{ℓ}, and φ_{ℓ}, which seems to be confirmed by practical experience in the case of two-dimensional plane flows, save for some cautionary asides which must be observed.

This hypothesis is the foundation upon which the treatment discussed here is to be built.

The essential factors which govern what the conditions of the flow shall be at reattachment are, thus, in view of what has just been said, the following:

(a) the internal fundamental factors describing the local behavior of the flow which reflect the type and extent of mixing that has occurred, as specified through means of the quantities φ and Θ, or which produce a change in position of the dividing streamline, as result of blowing or sucking.

(b) the external factors which act to change the curvature of the jet-like flow issuing from off the shoulder of the base during the whole period when mixing is taking place. That is to say, the factors in question act to control the direction that the jet flow acquires just before the moment of reattachment.

The most easily analyzed situation which obeys the stipulations resulting in a flow such as the one shown in Figure 1 is the case of a uniform two-dimensional plane flow, for which the upstream Mach number is M_o and the static pressure is p_o ahead of the step, and which, above all, admits of practically no boundary layer thickness in comparison with the height h of the rearward facing step at the point B. At this corner the previously undisturbed flow becomes detached and undergoes a change of direction of amount ψ_1. Because of this angular deviation, the external inviscid flow takes on new values, M_1 and p_1, for its local Mach number and pressure after negotiating the turn. Along the entire stretch of the isobaric zone BC the direction of this external inviscid flow remains fixed at the angle ψ_1.

Thanks to the stipulation of the "escape" hypothesis the subsequent angular deflection, $\Psi(M_1)$, experienced by the flow at the reattachment point can be determined by having recourse to results deduced from analysis of the mixing processes involved. Consequently, if the value of M_1 is known, then one may obtain the angle of the wall along which the flow courses after reattachment, ψ_2, as an incremental deflection from the direction of the upstream flow. Expressed analytically, this relationship between the flow directions may be written as

(1) $$\psi_2 = \psi_1 + \Psi(M_1) \qquad \text{where } \psi_1 < 0 \text{ for } p_1/p_o < 1.$$

Contrariwise, if the final direction ψ_2 is specified (the orientation of the downstream wall to which the stream reattaches is given) the above-written expression serves to determine the value of $\psi(M_1)$, and, consequently, the pressure in the inviscid flow downstream of the turn executed at the corner B may be found.

This very elementary flow configuration has already received the careful attention of a number of competent workers in the field, and, thus, this simple case may be used as a basic reference configuration. When more complicated cases come under examination in subsequent developments in which certain perturbating elements are allowed to intrude, it will be most illuminating to discuss and compare these less idealized situations with the flow pattern obtained in this basic model or reference example.

In illustration of what is meant by an internal fundamental factor influencing the reattachment, the state and extent of the boundary layer at the corner B may be cited. It will be shown how a thin turbulent boundary layer, that is not negligibly thin in comparison to the step-height h, will exert its influence upon the reattachment phenomena. Likewise, the importance of a temperature variation in the "dead-water" region will be discussed on theoretical grounds.

In addition, it will be demonstrated that a simple direct relation exists between the effects of blowing and the presence of a non-negligibly thin boundary layer.

The theoretical analysis of the effects produced by these internal fundamental factors will be supported by a collection of pertinent experimental data.

In the second part of this investigation, the influence on the reattachment of so-called "curvature factors" will be examined. It will be shown how, theoretically, the results obtained in the first part of the paper for the case of two-dimensional plane flow may be extended to apply to cases where the approaching stream is perturbed from the two-dimensional pattern by certain "curvature factors", which may arise because either:

(1) the forebody from which the flow separates is a body of revolution, or

(2) the upstream flow is not uniform but, before separation, it has been subjected to some form of perturbation caused by such things as a curved surface on the forebody, the presence of entropy gradients, or the persistent effects of pressure gradients.

The results of the theoretical derivations will be compared with observations made from appropriate experiments.

Even though the practical situations treated herein and the analogous experimental verifications have been limited to the case of turbulent flows, it is quite evident that one may convert the treatment presented, without any essential difficulty, to apply equally well when the boundary layer is laminar.

It is equally obvious that all the results discussed in this paper are directly applicable to the problem of determining base pressures under similar circumstances.

II. THEORETICAL AND EXPERIMENTAL INVESTIGATION OF THE INTERNAL FUNDAMENTAL FACTORS INVOLVED IN REATTACHMENT

First of all the Korst theory will be recapitulated and generalized, and then the reattachment-angle hypothesis will be introduced. These concepts are then to be applied to the basic reference configuration. After examining this primitive situation it is then shown how the introduction of various perturbing factors affects the results. It is observed that such perturbations can affect the conditions under which the mixing zone is formed and developed, so that, in turn, the modified mixing processes can alter the way the reattachment takes place.

The case of weak perturbations is handled by resort to a linearization technique. This approach leads to the establishment, for one thing, of a very simple but useful connection between the effect on reattachment of the boundary layer in the approaching flow and the effect produced by injection of fluid into the dead-water region by blowing.

These analytic derivations are followed and buttressed by carrying out a comparison with the results obtained from selected experiments.

II.1 - Calculational Approach

II.1.0 - The theoretical derivation of the "escape" criterion rests on the study of the nature of the flow in the mixing region. The ideas involved originated in the study of jets and are carried over here to good advantage. It is taken for granted in this analysis that the velocity profiles $\varphi(y;x)$ and the density profiles $\rho(y;x)$ exhibit similitude, i.e., they are independent of their x-position along the jet boundary. To be more precise about this statement the following mathematical formulation is assumed to hold true. Let a convenient reference length be selected, to be designated by the symbol δ. It is then premised that a position parameter can always be devised in the form

$$\eta_p = \frac{\delta}{x} \cdot f\left(\frac{x}{\delta}\right)$$

in which the function $f\left(\frac{x}{\delta}\right)$ remains bounded regardless of the size of x, which is considered positive. Furthermore, one may formally designate a non-dimensional normal coordinate in the flow to take the form

$$\eta = \frac{y}{\delta} \cdot \eta_p = \zeta \cdot \eta_p$$

When this is done, the similitude assumption mentioned above may be expressed unequivocally by the following universal functions describing the velocity and density profiles of interest:

$$\varphi = \varphi(\eta;\eta_p)$$

$$\text{and } \theta = \theta(\eta;\eta_p,u_1)$$

which are to obey the general Navier-Stokes equations and which are to be in suitable form for satisfying the boundary conditions pertinent to this present problem.

II.1.1 - Description of the Velocity Profile φ

At the origin of coordinates, i.e., at the location of the corner B, where x = 0, the velocity profile has the following character:

$\varphi = 0$ for $\eta \sim y < 0$

$\varphi = \varphi_1\left(\frac{y}{\delta}\right) = \varphi_1(\zeta)$ for $0 \le y \le \delta$ (the boundary layer is taken to be equivalent to the one produced in the approaching plane flow)

$\varphi = 1$ for $y > \delta$

where δ has been selected as the thickness of the boundary layer of the approaching flow at the location B.

At any location lying downstream of the corner B, for which the abscissa x takes on a positive value, the velocity profile has the following nature:

$\varphi(-\infty) = 0$ the dead-water region is entered

$\varphi(+\infty) = 1$ the external inviscid flow region is entered

An approximate solution for the velocity-profile function satisfying these conditions may be obtained with little effort provided the agreement is accepted that the strict Navier-Stokes equations may be relaxed and in their place the simplified differential expression given by S.I. Pai substituted. This less stringent requirement takes the form of

$$(1) \qquad \frac{\partial u}{\partial x} = \frac{e(x)}{u_1} \frac{\partial^2 u}{\partial y^2}.$$

In the case of turbulent mixing, which is the only situation to be considered here, the solution for the boundary layer profile development is found by inserting the expressions given above for the similar velocity profiles into Pai's differential equation. The result of carrying out the indicated quadrature is then found to be

$$(2) \qquad \varphi(\eta,\eta_p) = \frac{1}{2}\left[1 + \text{erf}\,(\eta - \eta_p)\right] + \frac{1}{\sqrt{\pi}} \int_{\eta - \eta_p}^{\eta} \varphi_1\left(\frac{\eta - \beta}{\eta_p}\right) e^{-\beta^2}\, d\beta$$

The location parameter η_p evidently depends on the mixing coefficient e(x), but if one is content to confine his attention only to that part of the flow which is sufficiently far removed from the origin it may be legitimately assumed that the location parameter is given simply by the ratio

$$\eta_p = \sigma\delta/x$$

which can be taken as sufficiently accurate for values of the abscissa which are large in comparison with the initial thickness of the boundary layer δ (i.e., one must be far enough downstream from the shoulder B to ensure that x/δ is in the order of 10 or greater). In this ratio the proportionality factor σ will be a function only of the external inviscid flow Mach number, M_1 (see Reference 1).

In the special case, which is a common and very important one, where the initial boundary layer thickness at the shoulder B is completely negligible in extent ($\delta \longrightarrow 0$) there is no reference length upon which to base the similarity development, and it is nonsensical to talk of a location parameter η_p. In this instance it is permissible to express the velocity profile φ simply as a function of a normal non-dimensionalized ordinate described by the affine relation

$$\eta = \sigma \frac{y}{x}$$

and, in consequence, one obtains a suitable solution for the similar profiles by inserting the value of $\eta_p = 0$ into Eq. (2), to find formally that in this case

$$\overline{\varphi}(\eta) = \frac{1}{2} (1 + \operatorname{erf} \eta) \qquad (2')$$

II.1.2 - Description of the Density Profile Θ

The choice of the density profile Θ will next be made in such a manner as to best provide for taking into account the thermal transport processes which occur in the mixing zone. This inclusion of the thermal aspects of the mixing will be accomplished by having recourse to the same hypotheses which have already been well substantiated in carrying out classical boundary layer investigations.

Let it be assumed for present purposes, therefore, that the total enthalpy is conserved. This assumption implies clearly that the enthalpy of the dead water is equal to the stagnation enthalpy of the exterior inviscid flow. If, furthermore, the convention is agreed to that the limiting velocity attained when the exterior inviscid flow is expanded into a vacuum is to be denoted by w_1, and if the ratio of the actual velocity to the value w_1 is then denoted by u_1, it follows, by appeal to the law of conservation of energy, that one then may relate the local velocity, as measured by the quantity $\varphi\, u_1$, and the density profiles through the expression

$$\Theta = \frac{\rho}{\rho_1} = \frac{T_1}{T} = \frac{1 - u_1^2}{1 - u_1^2\ \varphi^2} \tag{3}$$

If, on the other hand, the stagnation enthalpy H_m of the dead water happens to be different than the stagnation enthalpy of the exterior inviscid flow, then this deficiency may be expressly stated in the form

$$H_m = w_m^2\ H_i \qquad \text{where } w_m \neq 1.$$

Once again, though, by having recourse to the law of conservation of energy one may relate the local velocity and density profiles through use of the slightly altered expression

$$\Theta = \frac{1 - u_1^2}{w^2 - u_1^2\ \varphi^2}$$

where w denotes the local reduced limiting velocity attainable by a fluid particle travelling along a streamline imbedded in the mixing region. The connection between Θ and φ is not specified fully as yet in this instance since one must also stipulate the behavior of w as a function of the normal coordinate η of the profile.

This choice of the function $w^2(\eta)$ must be made in such a way as to assure that at the extremities of the profile the following compatibility conditions are obeyed

$$w^2(+\infty) = 1$$

$$w^2(-\infty) = w_m^2$$

while at the same time this enthalpy-deficiency must behave on a traverse across the mixing zone in a manner which is similar to the variation exhibited by the velocity profile $\varphi^2(\eta)$. It is permissible for present purposes, therefore, to take the enthalpy deficit to have the following form

$$w^2 = w_m^2 + \varphi^2(1 - w_m^2)$$

so that, in consequence, the density variation is seen to be expressible as

(4) $$\Theta = \frac{1 - u_1^2}{w_m^2 - \varphi^2.(u_1^2 + w_m^2 - 1)}$$

which thus constitutes a generalization of Eq. (3), which was only valid when the total enthalpy was constant.

It may be worthy of mention that the generalized expression for the density profile may be cast into a form exactly equivalent to Eq. (3) by use of a fictitious reference velocity U_1, defined by the equation

(4') $$U_1{}^2 = \frac{u_1{}^2 + w_m{}^2 - 1}{w_m^2} = 1 - \frac{T_1}{T_m}$$

from whence it follows that

$$\theta = \frac{1 - U_1^2}{1 - U_1^2 \varphi_1^2(\eta)} \tag{4''}$$

II.1.3 - Inward Shift of the Velocity and Density Profiles

The functions $\varphi(\eta)$ and $\theta(\eta)$, which have been defined in the manner just indicated to represent approximately the general behavior of the mixing processes, obviously do not obey the exact governing equations of the flow.

In order to satisfy the criterion of over-all conservation of momentum, the following improvements may be introduced. To do this it need merely be observed that one can always shift the origin of the y axis at any particular abscissa location in order to make the θ and φ profiles obey the imposed conservation of momentum condition. To show how the shift in axis is carried out in precise detail the following notation and procedures will be resorted to. Let $B\mathcal{X}$ represent the boundary of the isobaric jet in inviscid flow which emanates from the corner B after the turn is made at the corner producing the associated Mach number M_1. Likewise, let NN' denote a streamline in the external flow which is located at a sufficient distance outboard of the boundary $B\mathcal{X}$ so that the flow there is completely uniform and uninfluenced by the mixing, at least to the second order of approximation in δ/Y_N (See Fig. 3).

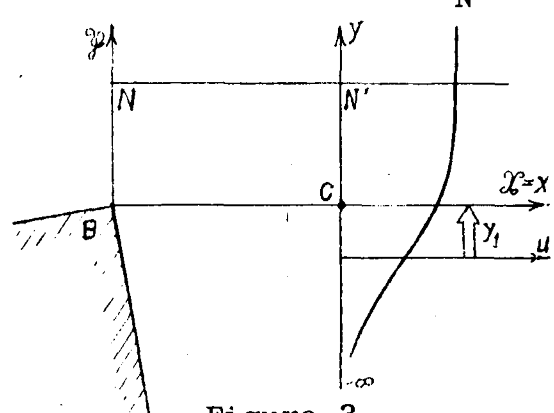


Figure 3
Inboard Shift of the Origin of the Velocity Profiles in the Mixing Zone

Now, the momentum computed at the control surface BN is expressible as

$$\int_{0}^{\mathscr{Y}_N} (\rho\, u^2)_{\mathscr{X}=0} \cdot d\mathscr{Y}$$

and this same amount of momentum must be accounted for at downstream locations, such as at the control surface CN'. Consequently one must have

$$\int_{0}^{\mathscr{Y}_N} \rho(0,\mathscr{Y})u^2(0,\mathscr{Y})d\mathscr{Y} = \int_{-\infty}^{\mathscr{Y}_N} \rho(\mathscr{X},\mathscr{Y})u^2(\mathscr{X},\mathscr{Y})d\mathscr{Y}$$

The left hand side is a known quantity, while the right hand side may be recast in terms of the local velocity and density profiles $\varphi(\eta)$ and $\Theta(\eta)$, where in the present instance the local ordinates are shifted downward by an amount $y_1(x)$, so that with

$$y = \mathscr{Y} + y_1(x)$$

$$\text{and} \quad \eta = \eta_p \frac{y}{\delta} = \zeta \cdot \eta_p$$

then the new non-dimensionalized ordinate becomes

$$\eta = \eta_p \cdot \frac{\mathscr{Y} + y_1(x)}{\delta} .$$

Consequently, by imposing the condition that there shall be conservation of momentum along the mixing region boundary, the downward shift with respect to the boundary $B\mathscr{X}$ that is required in the ordinates at each abscissa location is determined (see Figure 3) by the relation

$$\eta_1 = \eta_p \cdot \frac{y_1}{\delta}.$$

When the suggested substitutions and mathematical manipulations are carried out, one finds finally, that, upon passing to the limit $\mathscr{Y}_N \longrightarrow \infty$, the downward shifts required for preservation of momentum are given by the formula

$$(5)\quad \eta_1 = -\eta_p \int_0^1 \left(1 - \theta_1\, \varphi_1^2(\zeta)\right) d\zeta + \int_0^{+\infty} (1 - \theta\, \varphi^2)\, d\eta - \int_{-\infty}^{0} \theta\, \varphi^2\, d\eta$$

where θ_1 and φ_1 are the initial values of the density and velocity profiles at station B (where $\mathscr{X} = 0$) and where θ and φ are given by the expressions obtained previously as Eqs. (2) and (3). The details of this derivation are recapitulated in the appendix.

II.1.4 - Determination of Streamlines in the Mixing Zone

It is now possible to determine the streamline flow pattern in the mixing zone. Before considering any arbitrary streamline, it is worthwhile to fix attention on the particular streamline which represents the jet boundary (j), passing through the point J at the downstream abscissa location specified as $\mathscr{X}$. The desired relation for obtaining the downward shift for this jet boundary is obtained by working with the fluid fluxes passing the usual control surfaces. In the present case, if one focusses attention on the streamline NN' (across which no flux is carried) and the control surfaces BN and JN' (see Figure 4), it must result from continuity of the flow that

$$\int_{0}^{N} \rho(0,y)\; u(0,y)\; dy = \int_{J}^{N'} \rho(\mathscr{X},y)\; u(\mathscr{X},y)\; dy.$$

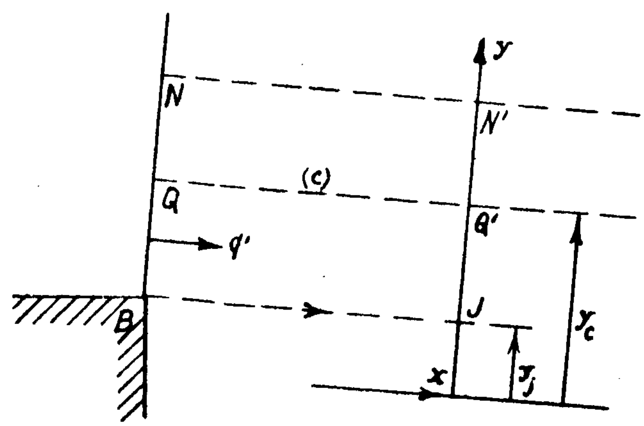


Figure 4

Orientation of Jet Boundary and Arbitrary Streamlines in the Mixing Zone

Upon insertion of the now familiar non-dimensionalized quantities η, η_p, ζ, and, by taking account of the result presented in Eq. (5), it may be shown, as is done in the appendix, that

$$(6) \qquad \int_{-\infty}^{\eta_j} \Theta\, \varphi\, d\eta = -\, \eta_p \int_{0}^{1} \Theta_1\, \varphi_1\, (1 - \varphi_1)\, d\zeta + \int_{-\infty}^{+\infty} \Theta\, \varphi (1 - \varphi)\, d\eta.$$

Consequently, this expression gives the value of the successive downward shifts in the jet boundary because it determines η_j as a function of η_p (i.e., y_j is determined, and, in turn, the velocity and density are determined as functions of x, inasmuch as $u_j = u_1\ \varphi(\eta_j)$ and $\rho_j = \rho_1\ \Theta(\eta_j)$).

In the case of an arbitrary streamline, the procedure continues as follows. Associated with each arbitrary streamline under discussion is a fluid flux, denoted by q', which is contained within a channel formed by the streamlines passing through J and Q'. Consequently a particular streamline of interest will be characterized by a non-dimensionalized flux parameter expressed as

$$\frac{q' \cdot \eta_p}{\rho_1 u_1 \delta} = \int_{\eta_j}^{\eta_c} \Theta \cdot \varphi \cdot d\eta \tag{7}$$

where η_j is the same shift in ordinate obtained above in Eq. (6). Thus, the location coordinate for the particular streamline (c) is obtainable as a function of the parameters η_p, $\frac{h}{\delta}$, and $C_q' = \frac{q'}{\rho_1 u_1 h}$.

II.2 - Application of the Escape Hypothesis to Some Simple Cases

II.2.1 - The Angular Deflection Occasioned by Reattachment

The idea back of the escape hypothesis has been outlined in the introduction. It is based upon the assumption that the stagnation pressure obtained by an isentropic compression of the fluid following the path of the discriminating streamline upstream of the reattachment must be equal to the pressure p_2 which the external inviscid fluid exhibits when it is subjected to an angular deflection of Ψ, to become aligned with the downstream constraining wall after reattachment.

Fixing attention on the flow along the discriminating streamline it is convenient to designate the velocity there by $K = \varphi(\eta_\ell)$ while the density on this discriminating streamline is clearly denoted by the analogous symbol $\Theta(\eta_\ell)$. Let M_1 signify as usual the Mach number of the external inviscid flow which is attained after the turn at the corner B but before the reattachment takes place. Based on these conventions it is then well known that the stagnation pressure attained when such a flow along the discriminating streamline is brought to rest is given by the expression

$$\frac{P_{i\ell}}{P_1} = \left(\frac{1}{1 - K^2 u_1^2}\right)^{\frac{\gamma}{\gamma - 1}}$$

while the isentropic compression * of the external inviscid flow, starting in the state where the static pressure and velocity are denoted by (p_1, u_1), and ending in the condition represented by (p_2, u_2), is given by the relation

$$\frac{p_2}{p_1} = \left(\frac{1 - u_2^2}{1 - u_1^2}\right)^{\frac{\gamma}{\gamma-1}}$$

Having set forth these explicit relations it follows immediately by application of the escape hypothesis (whereby it is stipulated that $P_{i\ell} = p_2$) that the following connection is established between the velocities under examination:

$$1 - K^2 u_1^2 = \frac{1 - u_1^2}{1 - u_2^2}$$

*Some investigators maintain that this recompression takes place by means of a shock wave, with the result that the turning exhibited by the $\Psi(M_1)$ curve during reattachment is reduced. The assumption made here, however, has the advantage of simplifying the subsequent calculations and it doesn't seem to introduce any noticeable change in the fundamental parameters influencing the flow.

or, what amounts to the same thing, K is given in terms of the local Mach numbers as

$$\boxed{M_2^2 = (1 - K^2)\ M_1^2}\ .$$

The recompression of the external inviscid flow from flow at speed M_1 to flow at speed M_2 may be determined by relying on the pre-tabulated information given by the Prandtl-Meyer law. This law is valid in the present case where a two-dimensional plane flow is under examination. The Prandtl-Meyer function is obtained by evaluation of the integral

$$P(M) = \int - \frac{1}{M} \frac{\sqrt{-1 + M^2}}{1 + \frac{\gamma - 1}{2} M^2}\, dM$$

as has been pointed out in Reference 3. Consequently, the angular deflection experienced by the dividing streamline upon reattachment is obtained from the tables as a difference of entries, where, specifically,

(9) $$\Psi = P(M_2) - P(M_1)$$

Upon elimination of the common Mach number, M_2, from the expressions now deduced as Eqs. (8) and (9), it follows that the angular deviation suffered by the discriminating stream at reattachment is

(10) $$\boxed{\Psi = P(M_1\sqrt{1 - K^2}) - P(M_1)}$$

where K represents the ratio $\varphi_{\ell} = u_{\ell}/u_1$. Of course, u_{ℓ} is the velocity attained by the flow along the discriminating streamline, while u_1 represents the velocity in the external inviscid flow where the Mach number is M_1. It has already been noted in Paragraph II.1 how the theory for the mixing processes permits one, in theory, to determine the value of the velocity ratio K.

This ratio is a function, evidently, of the external inviscid flow velocity, u_1, of the location of the discriminating streamline (ℓ) (i.e., of the fluid flux parameter denoted by C_q'), and of the initial conditions present at the corner B which characterize the formation and subsequent development of the mixing zone; these initial conditions are designated most conveniently by the parameter $\varphi_1(\zeta)$.

II.2.2 - The Basic Reference Flow

In order to illustrate the practical application of the methods discussed above for obtaining the velocity profiles in the mixing region and the angular deflection at reattachment, the situation encountered in the case of the basic reference flow configuration will be examined first. The basic reference flow configuration, as mentioned earlier, is the case presented by an approaching uniform two-dimensional plane flow which detaches from the rearward facing step without the complicating presence of any appreciable boundary layer at the corner. For this simplified version of the flow detachment it was pointed out earlier that the velocity distribution in the mixing region is given by the expression

$$\bar{\varphi}(\eta) = \frac{1}{2}\,(1 + \operatorname{erf}\,\eta)$$

where

$$\eta = \sigma \cdot \frac{y}{x}.$$

The formula for finding the downward shift of the velocity profile $\bar{\eta}_j$ is found in the present instance by letting $\eta_p = 0$ in Eq. (6), from which it follows that

$$(\bar{6}) \qquad \int_{-\infty}^{\bar{\eta}_j} \bar{\theta} \cdot \bar{\varphi}\, d\eta = \int_{-\infty}^{+\infty} \bar{\theta} \cdot \bar{\varphi}\,(1 - \bar{\varphi})\,d\eta .$$

The equation for any arbitrary streamline in the mixing region may then be found almost immediately by working with Eq. (7) in which the simplification is made now that $\frac{\eta_p}{\delta} = \frac{\sigma}{x}$.

In particular, the discriminating streamline (ℓ), represented by the parameter η_ℓ, will be linked directly to the amount of fluid flux, q, that is either injected or sucked out of the dead-water region. The location of the discriminating streamline is thus dependent upon imposition of the requirement that the mass of fluid in the dead-water region must remain constant, which may be symbolized by writing $q + q' = 0$.

If the flux coefficient is then defined by the relation

$$C_q = \frac{q}{\rho_1\, u_1\, h}$$

where $q > 0$ when blowing occurs
and $q < 0$ when sucking occurs

then one obtains the location coordinate of the discriminating stream from

$$(\bar{7}) \qquad -\frac{\sigma h}{x}\, C_q = \int_{\bar{\eta}_j}^{\bar{\eta}_\ell} \bar{\theta} \cdot \bar{\varphi} \cdot d\eta .$$

(a) Case of Reattachment when Neither Blowing Nor Sucking Takes Place

In accordance with the dictates of Eq. ($\overline{6}$) it is seen that the value of $\overline{\eta}_j$ depends solely on the parameter u_1 (or on the equivalent Mach number, M_1) provided one limits the discussion to the particular case for which $w_m \cong 1$. In consequence of this assumption, which amounts to saying that the limiting velocity for the streamlines in the mixing zone has a common value (is conserved), it follows that

$$\Theta = \frac{1 - u_1^2}{1 - u_1^2 \overline{\varphi}^2}$$

Under these stipulated conditions, one may deduce, therefore, that

$$\overline{K}(u_1) = \overline{\varphi}(\overline{\eta}_j)$$

and then through appeal to Eq. (10) the corresponding value of the angular deflection may be obtained, i.e., $\Psi(M_1)$.

In what follows this value of $\Psi(M_1)$ will be referred to as the required angular deflection for reattachment which applies in the case of the basic reference flow configuration. The variation of the functions $\overline{K}(M_1)$ and $\Psi(M_1)$ produced by a Mach number spread from 2 to 4.5 is shown in Figure 5.

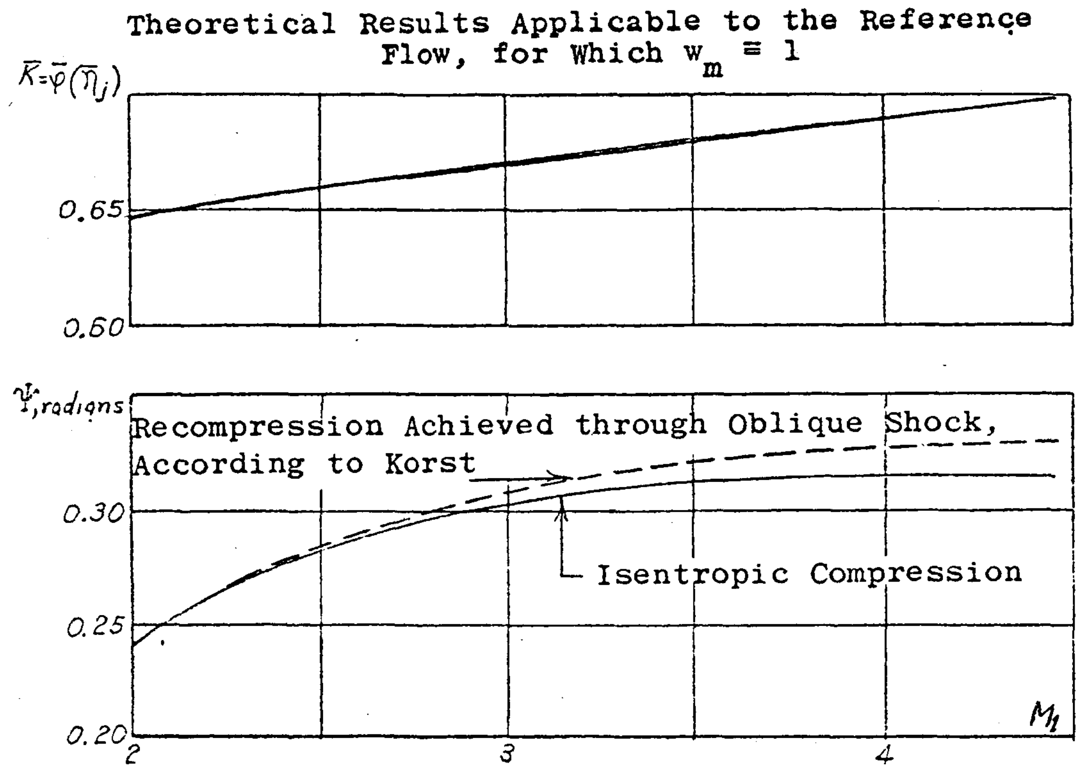


Figure 5

Required Angular Deflection for Reattachment, Ψ, and the Reduced Velocity Factor, $\bar{K}$, As Functions of the Upstream Mach Number, M_1

(b) Case of Reattachment in Presence of Blowing or Sucking, When $C_q \neq 0$

The flow is now reexamined in the case where blowing ($C_q > 0$) or sucking ($C_q < 0$) is taking place in the dead-water region. In this instance the location of the discriminating streamline is determined by evaluation of the non-dimensional ordinate $\bar{\eta}_\ell$ by use of Eq. (7). The numerical computation of $\bar{\eta}_\ell$ can only be carried out, however, provided one is supplied with the values of σ and x/h.

Now the mixing parameter σ has been studied in some detail by Görtler (see Reference 3) and by other investigators (see Reference 4). The results of these studies appear to indicate the applicability of the following empirical formula for evaluation of σ:

$$\sigma = 12 + 2.76\ M_1$$

In the choice of the value for x/h, on the contrary, it will be necessary to fall back on stipulation of a supplementary hypothesis. If the reattachment takes place within a reasonably small distance from the stagnation point R, it should not be greatly at variance with reality to admit, as Korst has suggested, that

$$\frac{BR}{h} = \frac{x}{h} = \frac{1}{\sin\ \psi_1} = \frac{1}{\sin\ \Psi}.$$

Unfortunately, experimental data [5] show unmistakably that the reattachment zone extends over a rather significant portion of the distance between B and R. Now, furthermore, Eq. (7) really is not applicable in the present case because it applies only to the constant-pressure portion of the mixing zone. Thus, the most sensible procedure is to make the assumption that

$$(11) \qquad \frac{x}{h} = \frac{\lambda_q}{\sin\ \Psi}$$

where λ_q is used to denote an empirical constant to be determined by experiment. Once this constant is decided upon, then $\overline{\eta}_\ell$ can be obtained by having recourse to Eq. (7) and, in consequence, the velocity ratio on the discriminating streamline is obtained as $K = \overline{\varphi}(\overline{\eta}_\ell)$. In addition, the angular deviation, now denoted as

$\Psi(M_1, C_q)$, may be obtained by use of Eq. (10). Typical values are given in Figure 6.

Theoretical Results Applicable to the Reference Flow, for Which $w_m \equiv 1$

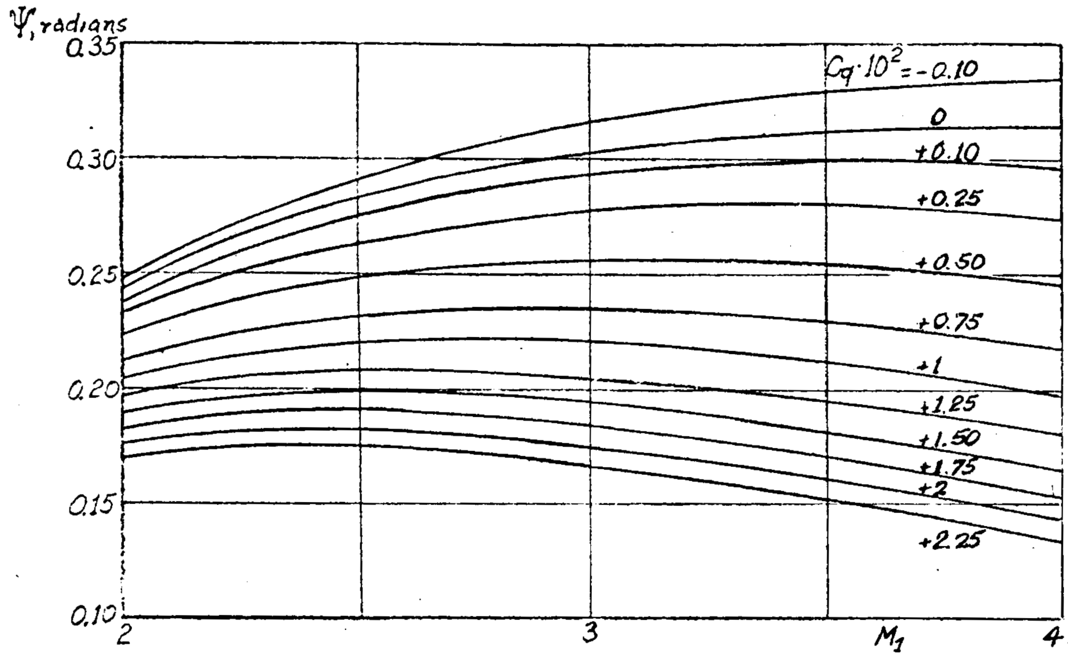


Figure 6

Required Angular Deflection for Reattachment, Showing the Effect of Blowing or Sucking

II.2.3 - Flows Differing Only Slightly from the Basic Reference Configuration

The required angular deflection for reattachment depends, in principle, solely upon the Mach number, M_1, and upon the velocity ratio at the discriminating streamline, K, according to the dictates of Eq. (10). A flow which differs only slightly from the basic reference flow configuration is now taken to mean a flow which is a linear perturbation from the reference flow. Thus, such a "neighboring" flow is any one which has a Mach number M_1 for the external inviscid flow and which exhibits a perturbing factor which operates to change the value of K by only a slight amount. By a slight amount is meant any change in the value of K (to be indicated by δK) such that the ensuing effect exerted upon the angular deflection at reattachment, Ψ, will be amenable to determination by linearization.

The perturbation in the required angular deflection may be formally written in the following way, on the basis of Eq. (10):

$$\delta\Psi = \frac{\partial \Psi}{\partial K} \cdot \delta K$$

and, by introduction of the expression for P(M), it follows that the sought expression for the angular deflection increment is

$$\boxed{\delta\Psi = \frac{K\delta K}{1 - K^2} \cdot \frac{1}{1 + \frac{\gamma-1}{2} M_1^2 (1-K^2)} \sqrt{M_1^2(1 - K^2) - 1}} \quad . \tag{12}$$

If the particular type of perturbation under consideration happens to be such that only the location parameter of the discriminating streamline, η_ℓ, is involved, while nothing else enters the picture to change the law that the velocity ratio $\overline{\varphi}(\eta)$ obeys, then one may write down an explicit relationship for the δK, as a function of $\delta\eta$. In fact the relation which is pertinent is just

$$\delta K = \frac{1}{\sqrt{\pi}} e^{-\overline{\eta}_j^2} \cdot \delta\eta \tag{13}$$

This case is exemplified by situations in which only a light blowing or sucking in the deadwater region is acting to perturb the flow, or if only a slight modification in the temperature is exerting its influence on the flow.

Application I - Case of Slight Blowing or Sucking

Upon taking the requirements of Eq. (11) into account, it is found by reference to Eq. (7) that the separation between the jet boundary and the discriminating streamline is given by the relation

$$\delta\eta = \bar{\eta}_{\ell} - \bar{\eta}_{j} = - \frac{\sigma}{\lambda_q} \cdot \frac{\sin \Psi}{\bar{\theta}_j \, \bar{\varphi}_j} C_q$$

where $\bar{\varphi}_j = \bar{K}$

and $$\bar{\theta}_j = \frac{1 - u_1^2}{1 - \bar{K}^2 u_1^2} = \left[1 + \frac{\gamma - 1}{2} M_1^2 (1 - \bar{K}^2) \right]^{-1}$$

Furthermore, the increment in the angular deflection $\delta\Psi$ is determined by recourse to Eqs. (12) and (13), which turns out to expressible, thus, as

$$\delta\Psi = - \Psi' \cdot \frac{\sigma \sin \Psi}{\lambda_q} \cdot C_q$$

wherein $$\Psi' = \frac{1}{\sqrt{\pi}} \cdot e^{-\bar{\eta}_j^2} \cdot \sqrt{\frac{M_1^2 (1 - \bar{K}^2) - 1}{1 - \bar{K}^2}} \, .$$

This relationship for the increment in the angular deflection is shown in Figure 7 for two possible values of the mixing parameter, σ.

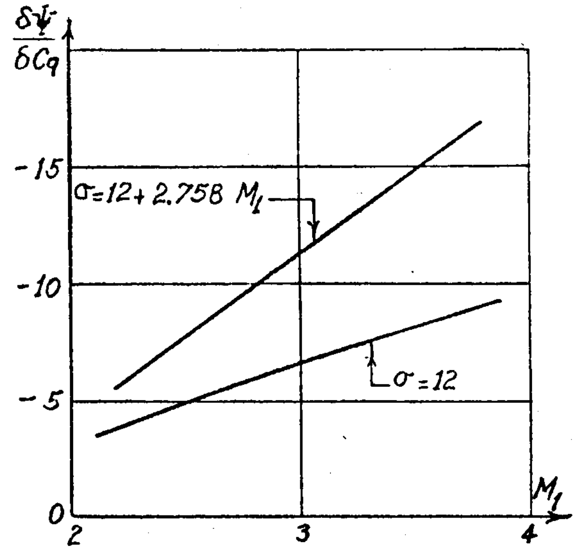


Figure 7

Increment in the Required Angular Deflection for Reattachment, Due to a Slight Amount of Blowing, for Two Different Kinds of Mixing Coefficient

Application I' - Momentum Additions

In the preceding case it has been tacitly assumed that the blowing ($C_q > 0$) or the sucking ($C_q < 0$) has been accomplished at very slow velocities, so that there is no need to take into account any transfer of momentum.

In the present instance, on the contrary, the case is to be examined in which an amount of momentum, denoted by J, is imparted to the flow in the absence of any mass transfer. The momentum is assumed to be transported across the base region ($x = 0$) confined to that portion for which $y < 0$, and is transmitted in a direction parallel to the Bx boundary.

Returning to considerations entirely analogous to those encountered in Section II.1.3, it is clear that if the conservation of momentum is premised, then one may write in this case that

$$J + \int_{0}^{N} \rho\, u^2\, dy = \int_{-\infty}^{N} \rho\, u^2 dy .$$

x = 0 station (x) station

If a momentum transport coefficient is now defined to have the form $C_\mu = \frac{2 \cdot J}{\rho_1 \, u_1^2 \, h}$, it is easy to see that one needs to add the term $\frac{\sigma}{2} \frac{h}{x} C_\mu$ to the right hand side of each of the expressions obtained previously as Eqs. (5) and (6). It is then found that the equation which defines the inward shift of the velocity profiles, η_j, may be written in the form

$$\int_{-\infty}^{\eta_j} \theta \cdot \varphi \cdot d\eta = \int_{-\infty}^{+\infty} \theta \, \varphi (1 - \varphi) \, dy + \frac{\sigma}{2} \frac{h}{x} C_\mu$$

where all other perturbations are being disregarded, so that $\eta_p = 0$. When this result is compared to the analogous expression obtained for the basic reference flow configuration, it is apparent that

$$\eta_j - \overline{\eta}_j = \delta\eta_j = \frac{\sigma}{2} \frac{h}{x} C_\mu \cdot \frac{1}{\overline{\theta} \, \overline{\varphi}}.$$

Having elicited this result for the streamline shift, it follows that the incremental change brought about in the value of the discriminating streamline velocity parameter, K, is given by the formula

$$\delta K = \frac{1}{\overline{\theta} \, \overline{K}} \cdot \frac{d\overline{\varphi}}{d\eta} \cdot \frac{\sigma \sin \Psi}{\lambda_q} \cdot C_\mu \cdot \frac{1}{2}$$

and, similarly, the incremental change in the required angular deflection occasioned by the introduction of the momentum is represented by

(15) $$\delta\Psi = \Psi' \cdot \frac{\sigma \sin \Psi}{\lambda_q} \cdot C_\mu \cdot \frac{1}{2} \, .$$

In the case where there is both injection of fluid ($C_q > 0$) as well as transport of momentum ($C_\mu > 0$), the formula given above as Eq. (14) should be altered to read

(14') $$\delta\Psi = \Psi' \cdot \frac{\sigma \sin \Psi}{\lambda_q} \left(\frac{1}{2} C_\mu - C_q \right).$$

NOTE:

In the event that the transport of momentum and of fluid takes place at a constant velocity V_j, then in this particular case it is permissible to write

$$C_\mu = 2 \, C_q \frac{V_j}{u_1}$$

and the incremental required angular deflection is determined by the expression

$$\delta\Psi = - \Psi' \cdot \frac{\sigma \sin \Psi}{\lambda_q} \cdot C_q \left(1 - \frac{V_j}{u_1} \right).$$

Inasmuch as C_q is itself proportional to V_j, it is evident from the above formula, then, that the effect on $\delta\Psi$ of a combined momentum and fluid injection which takes place at a progressively increasing rate will eventually arrive at a maximum amount of deflection, which will then decrease thereafter, as is confirmed by experiment. If, furthermore, the density ρ_j at the injection is maintained constant, then the maximum deflection is attained when $V_j = u_1/2$.

Application II - Alteration in Temperature Level of the Fluid in the Dead-Water Region

In this example it is assumed that the enthalpy of the fluid in the dead-water region is regulated by some device so that $w_m = 1 + \delta w_m$. Within the framework of the hypotheses that have been agreed upon it is evident that such an enthalpy perturbation will not have any effect on $\overline{\varphi}$, but it will only influence the value of $\overline{\Theta}$. Now it has been shown above in a formal manner that $\overline{\Theta}$ does not change its mathematical expression when the real velocity u_1^2 is replaced by the fictitious velocity $U_1^2 = \dfrac{u_1^2 + w_m^2 - 1}{w_m^2}$.

Consequently, Eq. (6), which is the expression arrived at previously for the jet boundary location will now provide the sought value of η_j as $\overline{\eta}_j\ (U_1)$ here, instead of $\overline{\eta}_j(u_1)$. Furthermore, the velocity ratio on the discriminating streamline will be found, then, as

$$K = \overline{\varphi}(\eta_j) = \overline{K}(U_1).$$

Besides, the imposition of the "escape" hypothesis, in this case where the limit velocity of the flow is no longer unity but is now taken to be given by $w^2 + K^2\ (1 - w_m^2)$, leads to an alteration in the result given in Section II.2.1 for the isentropic stagnation pressure for the discriminating streamline. Under present stipulations this pressure is obtained from the formula

$$\frac{P_{i\ell}}{P_1} = \left(\frac{w_m^2\ (1 - K^2) + K^2}{w_m^2\ (1 - K^2) + K^2 - K^2\ u_1^2} \right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{1 - u_2^2}{1 - u_1^2} \right)^{\frac{\gamma}{\gamma-1}}.$$

and one regains a result for the downstream Mach number, M_2, which is formally the same as the relation arrived at earlier as Eq. (8), except that in the present instance K must be replaced by K_1, i.e.,

$$(8') \quad \begin{cases} K_1^2 = K^2 \cdot \dfrac{1}{K^2 + w_m^2(1 - K^2)} \\ \\ M_2^2 = (1 - K_1^2)\, M_1^2. \end{cases}$$

In summary, it has been shown that if the limit velocity of the flow in the dead-water region is envisioned as linked to the limit velocity of the mixing zone (i.e., this limit velocity is the one attainable by expanding down to vacuum pressure the flow in the mixing zone) then this new incremental value for the limit velocity of the flow in the dead-water region implies that the following alterations must be incorporated into the calculation of the required angular deflection, Ψ:

1. The value of K is obtained from Fig. 5 by entering the curve with the value $U_1 = \sqrt{u_1^2 + w_m^2 - 1/w_m}$ in place of u_1. Thus, one finds that $K = \bar{K} + \delta_1 K$.

2. The required angular deflection at reattachment is calculated by having recourse to Eq. 10, wherein K is replaced by the new expression obtained from Eq. (8'). This amounts to introduction of a second correction applied to K, which may be designated as $\delta_2 K$.

Provided the value of w_m is not significantly different from unity, so that $w_m = 1 + \delta w_m$, with δw_m small, then it is found that

$$\delta_1 K = \delta\, w_m \cdot \frac{1 - u_1^2}{u_1} \frac{d\bar{K}}{du_1}$$

and

$$\delta_2 K = -\,\delta\, w_m\, \bar{K}\,(1 - \bar{K}^2).$$

Thus, one may then legitimately employ the differential expression

$$\delta\Psi_w = \delta w_m \cdot K_w' \cdot \frac{d\Psi}{d\bar{K}}$$

where

$$K_w' = \frac{1 - u_1^2}{u_1} \cdot \frac{d\bar{K}}{du_1} - \bar{K}\,(1 - \bar{K}^2).$$

The value of $\frac{d\Psi}{d\bar{K}}$ is given as before by Eq. (12), while the coefficient K_w' is depicted graphically in Figure 8.

Theoretical Results Applicable to the Reference Flow, for Which $w_m \equiv 1$

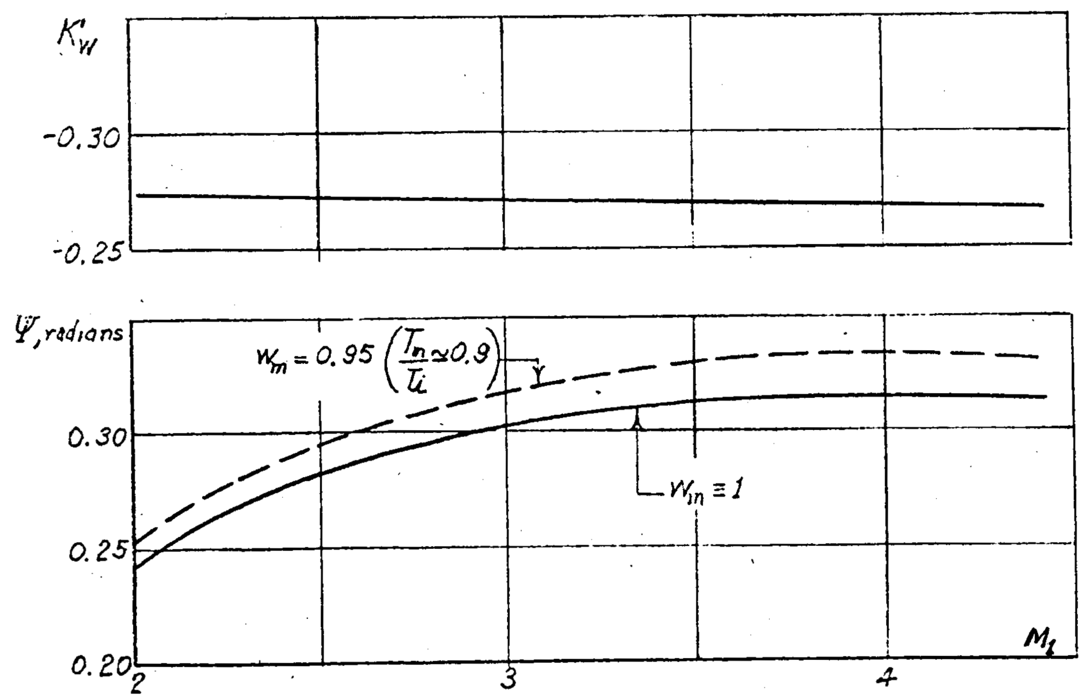


Figure 8

Effect on the Reattachment Flow of Alteration in the Temperature Level of the Fluid in the Deadwater Region

Numerical Example:

For purposes of numerical illustration let the following specific case be examined. The stagnation temperature in the exterior inviscid flow is taken to be $T_{i_1} = 300°K$, while the temperature of the dead-water region is kept at the value $T_m = 270°$.

Thus, $$\delta w_m = - \frac{300 - 270}{2 \times 300} = - 0.05.$$

Let the Mach number of the flow after turning the corner be taken as $M_1 = 2.63$. Then the value of u_1^2 is 0.584, and it follows immediately that

$$K - 0.663, \quad \frac{d\Psi}{d\bar{K}} = 1.13, \text{ and } \frac{d\bar{K}}{du_1} = 0.18.$$

Consequently, $$K_w' = \frac{(0.416)(0.18)}{0.764} - (0.663)(0.56) = 0.098 - 0.371 = -0.273$$

$$\text{and } \delta\Psi_w = (0.05)(0.273)(1.13) = + 0.0154 \text{ radian}$$

from which the required angular deviation at reattachment may be deduced to be $\bar{\Psi} = 0.29$.

It is evident from this example that the first increment, $\delta_1 K$, is relatively small in comparison with the second one, $\delta_2 K$. It is also evident that a quite moderate shift in the temperature at which the mixing occurs results in increments in the required angular deviation at reattachment, Ψ, which are not at all ignorable according to the graph of Figure 8.

The influence exerted upon such separated flows of the presence of an appreciable boundary layer will be treated next. This sort of flow condition will also be illustrated with a numerical application later on.

II.3 - Effect on Reattachment of the Presence of a Boundary Layer

In any real flow there always will exist a boundary layer lying next to the wall that is confining the flow. For present purposes the nature of this boundary layer will be specified by selecting the velocity profile and the temperature profile which is achieved in the development of the boundary layer just as it reaches the shoulder B immediately before separation occurs. These profiles may be designated symbolically as

$$\frac{u'}{u_o} = \varphi_o\left(\frac{y'}{\delta'}\right) \text{ and } \frac{\rho'}{\rho_o} = \frac{T_o}{T'} = \Theta_o\left(\frac{y'}{\delta'}\right)$$

where δ' represents the real physical thickness of the boundary layer; i.e., it is granted that $\varphi_o = 1$ and $\Theta_o = 1$ for all $y > \delta'$. Of course, as usual, the symbols M_o, u_o, and T_o represent the Mach number, the velocity, and the temperature, respectively, of the inviscid flow encountered at the shoulder B (see Figure 9).

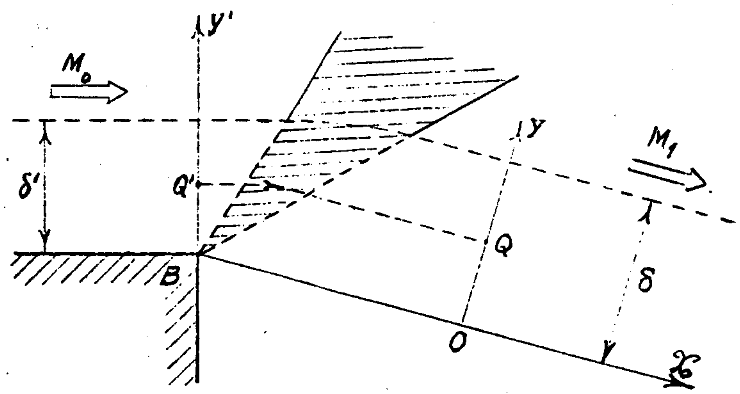


Figure 9

Boundary Layer Flow Pattern for Rapid Expansion at Corner: Two-Dimensional (Plane) Case

As the flow passes the corner B it undergoes a very rapid expansion (which occurs instantly at the point B itself, in the absence of any boundary layer). This expansion transforms the exterior inviscid flow from the conditions (M_o, p_o) into a new set of conditions (M_1, p_1). In order to progress in the study of the detachment and reattachment processes it is necessary, first of all, to find out how such an expansion reacts on the boundary layer. After determining the effect on the boundary layer itself, then, in turn, one can proceed to determine what perturbations are induced by the boundary layer upon the nature of the velocity and temperature profiles to be encountered in the mixing zone just before reattachment.

II .3.1 - Effect of a Rapid Expansion upon the Boundary Layer Development

The analytic expression for the expansion taking place at the shoulder B can best be denoted by use of the pressure ratio p_1/p_o. Inasmuch as this expansion process is an isentropic one in the exterior inviscid flow, the representative pressure ratio is connected with the other pertinent flow parameters by the following set of relations:

$$\frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_o^2} = \frac{T_o}{T_1} = \frac{a_o^2}{a_1^2} = \left(\frac{p_o}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \Delta.$$

After the expansion takes place, the inviscid external flow attains a new Mach number, denoted by M_1, and it assumes the direction of the line $B\mathscr{X}$, which is determined by the well-known Prandtl-Meyer relation for deflection of a supersonic flow through an expansion.

Keeping in mind these characteristics of the external inviscid flow, it is reasonable to expect that the boundary layer will obey the following rules as it undergoes the expansion at the shoulder B:

A. - For a very short distance downstream of the shoulder B, of the order of magnitude of δ', one will find that the velocity vectors lying along every streamline throughout a normal section Oy will be aligned in the same direction, which will be the direction of $B\mathscr{X}$, and that the static pressure will be constant there and equal to p_1.

B. - If Q' and Q are used to denote two points lying on the same streamline and so situated that one lies upstream of the expansion; i.e., so that the first (Q') lies on the ordinate axis By' and the second (Q) lies on the rotated axis Oy, then the total enthalpy and the entropy are conserved while the flow passes from Q' to Q. As thus described, the points Q' and Q are said to be homologous points.

This hypothesis is equivalent to acceptance of the fact that when the flow passes from Q' to Q it undergoes a quasi-discontinuous change in which the effects of the pressures are infinitely greater in importance than the effects of viscosity or thermal diffusion.

Having made these stipulations A and B it is easy to see what their immediate consequences are as far as linking the flow parameters which hold upstream and downstream of the expansion. Let the upstream parameters be denoted by primes, so that at the homologous point Q' to Q the quantities of interest are y', u', a', and T' (while at Q itself the corresponding quantities are denoted simply by y, u, a, and T), and also take it for granted that the limit velocity to which the flow can be expanded is to be denoted by unity. Having agreed upon these conventions, it then follows that

$$(2) \qquad \frac{a_1^2}{a^2} = \frac{T'}{T} = \Delta$$

$$(3) \begin{cases} \text{and } u^2 + \dfrac{2}{\gamma - 1} \cdot a^2 = w^2 \quad ; \quad u_1^2 + \dfrac{2}{\gamma - 1} \cdot a_1^2 = 1 \\ (u')^2 + \dfrac{2}{\gamma - 1} \cdot (a')^2 = w^2; \quad u_o^2 + \dfrac{2}{\gamma - 1} \cdot a_o^2 = 1. \end{cases}$$

Upon looking back at Eq. (1) and taking into account Eq. (2) it becomes clear that

$$(4) \qquad \Theta_o(y') = \frac{T_o}{T'} = \frac{T_1}{T} = \Theta_1(y).$$

Hence, at any two homologous points, such as Q' and Q, the density function Θ exhibits the same value.

A simple manipulation of the equations given above results in the establishment of the following connection between $\varphi_1(y)$ and $\varphi_o(y')$:

$$(5) \qquad \varphi_1(y) = \sqrt{\frac{\varphi_o^2(y') + w^2 \mathcal{B}^2}{1 + \mathcal{B}^2}}$$

where $\quad \mathcal{B}^2 = \dfrac{\Delta - 1}{u_o^2} = \dfrac{M_1^2}{M_o^2} - 1.$

It remains to establish a point-to-point correspondence between the ordinate values (y,y') of homologous locations. This correspondence in position may be obtained by falling back on the continuity theorem, which requires that

$$\rho \; u \; dy = \rho' \; u' \; dy'$$

so that, it follows, one must take

$$dy = \Delta^{\frac{\gamma+1}{2(\gamma-1)}} \frac{\varphi_o}{\sqrt{\varphi_o^2 + w^2 \theta^2}} \, dy'.$$

For convenience sake, let the following notation be introduced

$$(6) \quad \begin{cases} \zeta = \dfrac{y}{\delta} \text{ and } \zeta' = \dfrac{y'}{\delta'} \\ \text{and} \quad Z(\zeta') = \displaystyle\int_o^{\zeta'} \frac{\varphi_o \; d\zeta'}{\sqrt{\varphi_o^2 + w^2 \theta^2}} \end{cases}$$

then it follows that

$$(7) \quad \begin{cases} \zeta = \dfrac{Z(\zeta')}{Z(1)} \\ \text{and } \dfrac{\delta}{\delta'} = Z(1) \cdot \Delta^{\frac{\gamma-1}{2(\gamma-1)}} \end{cases}$$

and it may be seen, in addition, that

$$\Delta^{\frac{\gamma+1}{2(\gamma-1)}} = \frac{M_1}{M_o} \cdot \frac{\Sigma(M_1)}{\Sigma(M_o)}$$

where the function $\Sigma(M)$ is used to denote the area ratio A/A_o, where A is the cross-sectioned area of a stream-tube, and where the area ratio in question is the one obtained by an isentropic expansion resulting in the Mach number M.

Let the special case now be examined in which the limit velocity w attains the value unity. That is to say, let attention be focused on the special case for which

$$\Theta_o(\zeta') = \frac{1 - u_o^2}{1 - u_o^2 \varphi_o(\zeta')} = \frac{1 - u_1^2}{1 - u_1^2 \varphi_1^2(\zeta)}$$

The integral given for evaluating Z in Eq. (6) of Section II.3.1 may thus be simplified to

$$Z(\zeta') = \int_o^{\zeta'} \frac{\varphi_o(\zeta')\, d\zeta'}{\sqrt{\Theta^2 + \varphi_o^2}} .$$

The corresponding momentum thicknesses which pertain to the flow upstream and downstream of the expansion may then be evaluated from the expressions

$$\frac{\delta_2'}{\delta'} = \int_o^1 \frac{1 - u_o^2}{1 - u_o^2 \varphi_o^2} \varphi_o (1 - \varphi_o)\, d\zeta'$$

and $$\frac{\delta_2}{\delta} = \int_o^1 \frac{1 - u_1^2}{1 - u_1^2 \varphi_1^2} \varphi_1 (1 - \varphi_1)\, d\zeta .$$

It may be remarked that the following alternate expressions hold for the terms appearing in the above expressions for the momentum thicknesses:

$$\frac{1 - u_o^2}{1 - u_o^2 \ \varphi_o^2} = \frac{1 - u_1^2}{1 - u_1^2 \ \varphi_1^2}$$

$$d\zeta = d\zeta' \cdot \frac{1}{Z(1)} \cdot \frac{\varphi_o}{\sqrt{\wp^2 + \varphi_o^2}} = d\zeta' \cdot \frac{\varphi_o}{\varphi_1} \cdot \frac{1}{Z(1)} \cdot \frac{1}{\sqrt{1 + \wp^2}}$$

and $\frac{\delta}{\delta'} = \frac{\sum (M_1)}{\sum (M_o)} \cdot \frac{M_1}{M_o} \cdot Z(1).$

Consequently, it follows that the momentum thickness ratio is given by

$$\frac{\delta_2}{\delta_2'} = \frac{\sum (M_1)}{\sum (M_o)} \frac{\int_o^1 \frac{\varphi_o(1 - \varphi_1)}{1 - u_o^2 \varphi_o^2} d\zeta'}{\int_o^1 \frac{\varphi_o(1 - \varphi_o)}{1 - u_o^2 \varphi_o^2} d\zeta'}$$

where the shorthand notation has been used that

$$\varphi_1(\zeta') = \sqrt{\frac{\varphi_o^2 (\zeta') + \wp^2}{1 + \wp^2}}$$

When the expansion is of only moderate degree it is readily appreciated that the right hand side of the expression for the momentum thickness ratio will not differ greatly from the function $\Sigma(M_1)/\Sigma(M_o)$. Consequently, the quotient of the two integrals which multiply $\Sigma(M_1)/\Sigma(M_o)$ may be treated as a correction factor, designated as A. The way this correction factor, A, behaves as a function of the expansion, represented by Φ, is shown graphically in Figure 10.

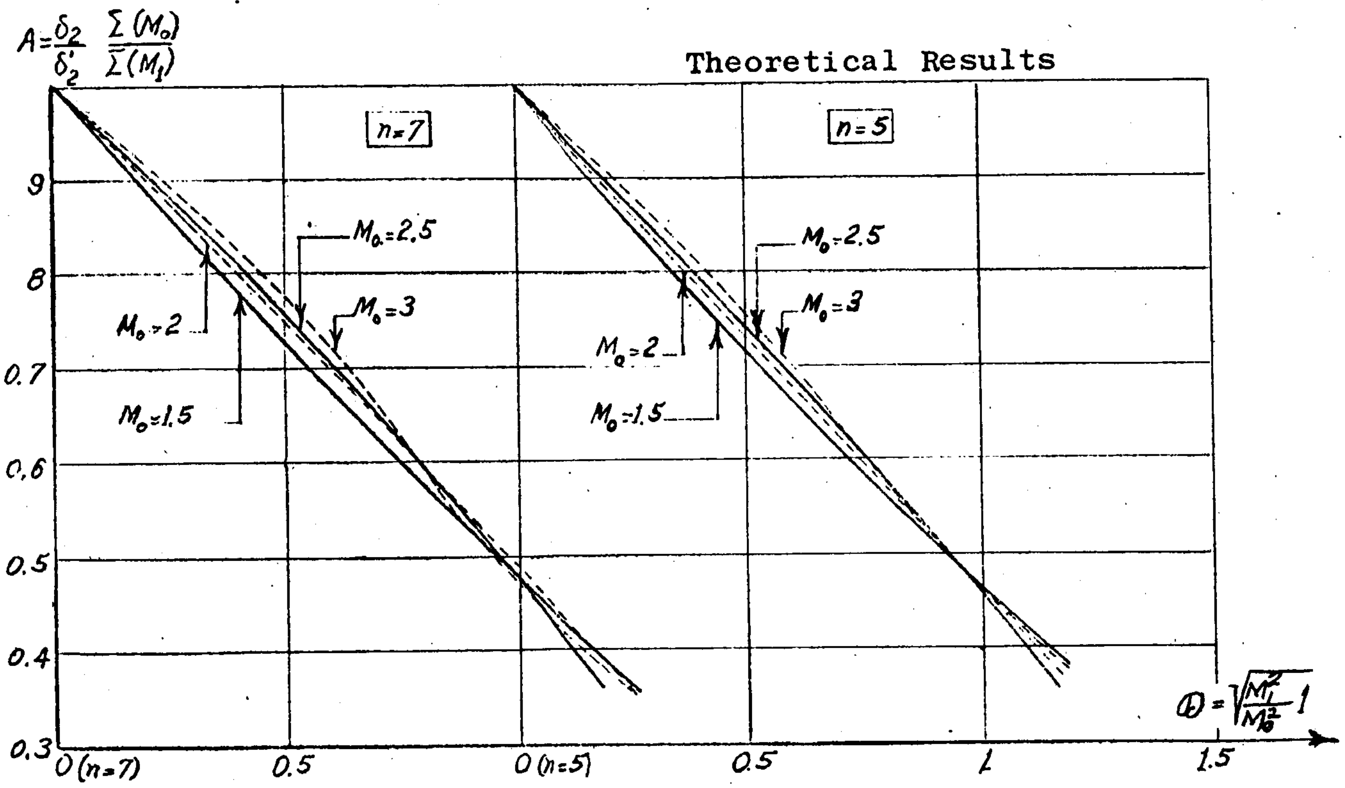


Figure 10
The Correction Factor, A, As Function of the Degree of Expansion, Φ.

II.3.2 - Determination of the Conditions at Reattachment

The calculation of the conditions at reattachment may be accomplished by going back and recalling some of the results already obtained in the preceding sections. The velocity profile in the mixing zone is given by Eq. (2), found in Section II.1.0, in the form

$$\varphi = \varphi\left[\eta,\ \eta_p,\ \varphi_1(\zeta)\right].$$

It has already been indicated how one obtains the initial velocity profile denoted by $\varphi_1(\zeta)$. The parameter η_p is the quantity which governs the effect on the velocity profile of the presence of the boundary layer, where

$$\eta_p = \frac{\sigma\delta}{x\ f(x/\delta)}$$

It is not necessary to pay any attention here to any other condition than that pertaining to the limiting value of f (x/δ), which is unity, which is for all practical purposes the value obtained when x/δ is sufficiently large.

Although the streamwise location, x, which is to be used to represent the distance corresponding to the reattachment point is not really known beforehand, nevertheless it is a practical procedure to make the selection, as was done in Section II.2.2, that

$$\frac{x}{h} = \frac{\lambda_\delta}{\sin \Psi}$$

from which it follows that

(8) $$\eta_p = \frac{\sigma}{\lambda_\delta} \cdot \frac{\delta}{h} \cdot \sin \Psi$$

where λ_δ is a numerical constant to be determined from experiment. This numerical constant is the proportional part of the distance from B to R which represents the fictitious length along which constant pressure mixing has occurred.

The downward shift in the jet boundary as represented by the quantity η_j may be determined by referring back to Eq. (6) of Section II.1.4. After obtaining η_j it is then a simple matter to evaluate $K = \varphi(\eta_j)$ by use of Eq. (2) of Section II.1.0. On the basis of this velocity ratio one may then go on to obtain the value of the required angular deflection at reattachment, Ψ, by means of Eq. (10) of Section II.1.1.

These outlined steps in the calculation may be carried through in a straightforward way but at the expense of considerable computational labor. The whole process may be considerably simplified, however, if one merely linearizes the equations.

With this object in mind, one may now premise that every quantity of order $\eta_p^{\ 2}$ may be neglected, and taking the basic reference flow configuration, represented by $(\overline{\varphi}(\eta), \overline{\eta}_j)$, as the norm from which perturbational variations are considered to start, it may be stated that

$$\varphi(\eta) = \overline{\varphi}(\eta) + \eta_p \cdot \varphi' + \mathcal{O}(\eta_p^{\ 2})$$

and

$$\eta_j = \overline{\eta}_j + \eta_p \cdot \eta' .$$

Upon differentiation with respect to η_p of the expressions for the velocity ratio φ, as given in Eq. (2) of Section II.1.2, and of the expression for the streamline locations as given in Eq. (6) of Section II.1.4, the following pertinent differential equations are obtained (when evaluation is made for the specific case of the basic reference configuration, for which $\eta_j = \overline{\eta}_j$ and $\overline{\varphi}(\eta_j) = \overline{K}$). The details are given in Appendix 1, but the significant results are that

$$\varphi' = \frac{e^{-\overline{\eta}_j^{\,2}}}{\sqrt{\pi}} \cdot \frac{\delta_{1_i}}{\delta} = - \frac{\delta_{1_i}}{\delta} \cdot \left(\frac{d\overline{\varphi}}{d\eta}\right)_{\eta=\overline{\eta}_j}$$

and $$\eta' = \frac{d\eta_j}{d\eta_p} = \frac{\delta_{1_i}}{\delta} - \frac{\delta_2}{\delta} \cdot \frac{1 - u_1^2 \overline{K}^2}{\overline{K}(1 - u_1^2)}$$

where $$\frac{\delta_{1_i}}{\delta} = \int_0^1 \left[1 - \varphi_1(\zeta)\right] d\zeta$$

and $$\frac{\delta_2}{\delta} = \int_0^1 \frac{\rho}{\rho_1} \frac{u}{u_1}\left(1 - \frac{u}{u_1}\right) d\zeta = \int_0^1 \theta_1 \, \varphi_1 (1 - \varphi_1) d\zeta$$

where the δ_{1_i} and δ_2 are the conventional boundary layer thicknesses of the approaching flow.

From these differential coefficients it follows that the increment in φ is given by

$$\varphi(\eta_j) - \overline{\varphi}(\overline{\eta}_j) = \delta K = \eta_p \left[\varphi' + \eta' \frac{d\overline{\varphi}}{d\eta}\right]$$

$$\text{or} \quad \delta K = - \eta_p \frac{\delta_2}{\delta} \frac{1 - u_1^2 \ \overline{K}^2}{\overline{K}(1 - u_1^2)} \frac{e^{-\eta_j^2}}{\sqrt{\pi}}.$$

Referring back to the expression obtained as Eq. (12) of Section II.2.3 for the increment in the required deflection, $\delta\Psi$, one now obtains a value which applies in the present case, when a boundary layer is perturbing the flow, which has the form

$$(9) \quad \begin{cases} \delta\Psi = \Psi\left(M_1, \dfrac{\delta_2}{h}\right) - \Psi(M_1) = \Psi' \cdot \dfrac{\delta_2}{h} \cdot \dfrac{\sigma \sin \Psi}{\lambda_\delta} \\ \text{where} \quad \Psi' = - \dfrac{1}{\sqrt{\pi}} e^{-\eta_j^2} \sqrt{\dfrac{M_1^2 (1 - \overline{K}^2) - 1}{1 - \overline{K}^2}}. \end{cases}$$

Upon comparing this expression with the one presented as Eq. (14) in Section II.2.3 a very remarkable theoretical observation comes to light, namely, that

$$\frac{\partial \Psi}{\partial C_q} = \frac{\partial \Psi}{\partial \frac{\delta_2}{h}} \cdot \frac{\lambda_\delta}{\lambda_q}$$

with the mere restriction that the ratio $\lambda_\delta / \lambda_q$ must not equal unity. The interpretation of this result is the following: If the boundary layer in the approaching flow is characterized by a momentum thickness designated by δ_2, then the effect of this boundary layer on the required angular deflection at reattachment is exactly equivalent to the effect of a mass injection by blowing into the dead-water region of amount $q = \rho_1 u_1 \delta_2$.

II.4 - Comparison with Experiment

II.4.1 - The Experimental Set-Up

The flow arrangement used in the experiments is represented schematically in Figure 11, while a more detailed description of the apparatus is fully described in Reference 5.

Sliding members, as shown, were able to be positioned along the dividing wall of a half-tunnel in which the Mach number could be selected as 2 or 3 depending on the nozzle block. With such a set-up, utilizing wedges which were movable and which were constructed with a series of different wedge angles, it was possible to change the reattachment angle of the downstream wall, ψ_2, to fit the experimenter's preference. Since the wedge is moveable, the effective height, h, of the backward facing step is also variable at the experimenter's will (see Figure 11).

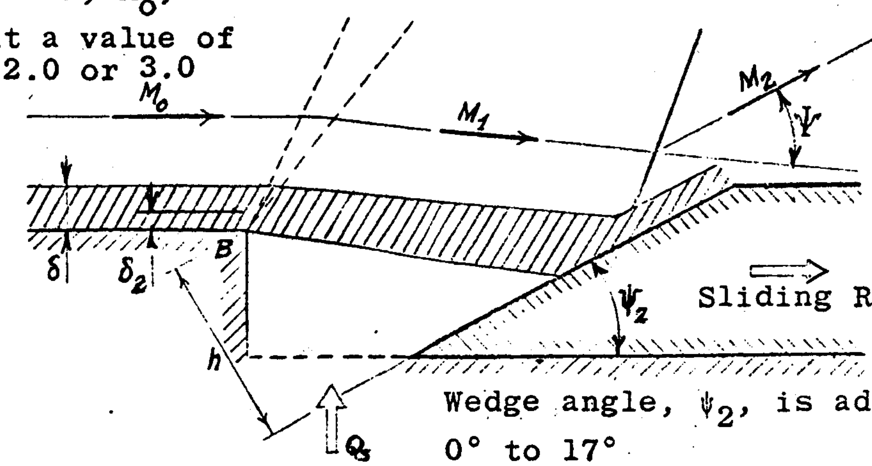


Figure 11

Schematic of Experimental Set-Up for Studying Reattachment Phenomena in Two-Dimensional (Plane) Flow

Inasmuch as the experimentally realizable values of M_o and ψ_2 provided by the physical apparatus were confined within moderate limits of variation, it turns out that the obtainable values of the inviscid flow Mach number, M_1, downstream of the corner, lie within the range of 2 to 4.25.

For each of the geometric configurations examined, the effects of blowing and sucking were observed by varying the amount of fluid flux introduced into the dead-water region. The effects produced by having different thicknesses of the boundary layer approaching the rearward facing step were studied, meanwhile, merely in a restricted number of typical cases. The variation in the ratio of the boundary layer thickness at the shoulder B with respect to the effective height of the rearward facing step, δ_2/h, was actually obtained by variation of the step-height through fore- and aft-movement of the sliding wedge.

II.4.2 - Effects of Fluid Injection into the Dead-Water Region

The primary objective of the investigation being reported here was study of the reattachment behavior of the separated flow, as affected by relatively weak values of the injected flux parameter, C_q. By keeping the value of this parameter under 0.02, the momentum transported from the secondary stream as it is injected into the dead-water region is practically inconsequential, so that this situation corresponds closely to the stipulations imposed by the theory developed above.

Reduction of the experimental data obtained during the course of the experiments which covered the range of parameters mentioned in the preceding Section II.4.1 has permitted the construction of the family of very significant curves, relating the reattachment angle ψ to the flux parameter C_q, as represented in Figures 12 and 13. In all, data have been obtained for six discrete Mach numbers, M_1, ranging from 2.15 to 3.70. Within the range of conditions met during these experiments the corresponding values of the ratio of the boundary layer thickness at the shoulder B to the backward facing step height, δ_2/h, varied over the range from 0.014 to 0.021.

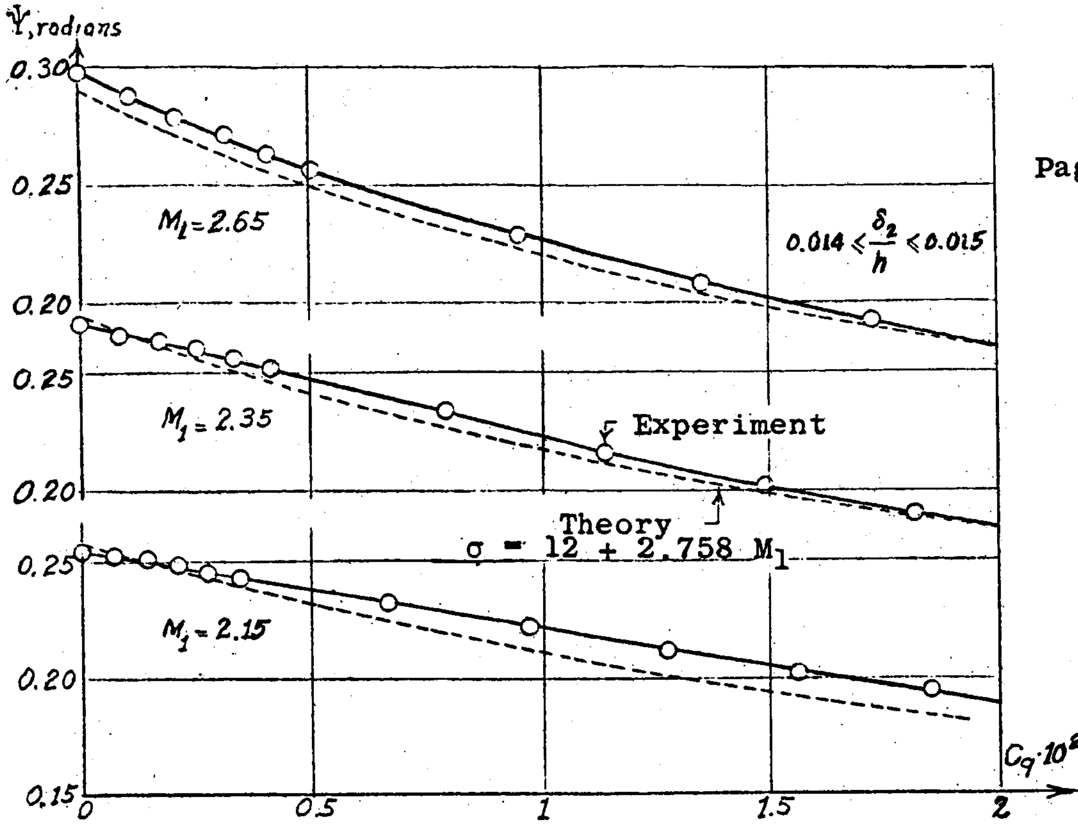


Figure 12

Effect of Blowing on Reattachment Angle - Comparison of Theory with Experiment for $2.15 \leq M_1 \leq 2.65$

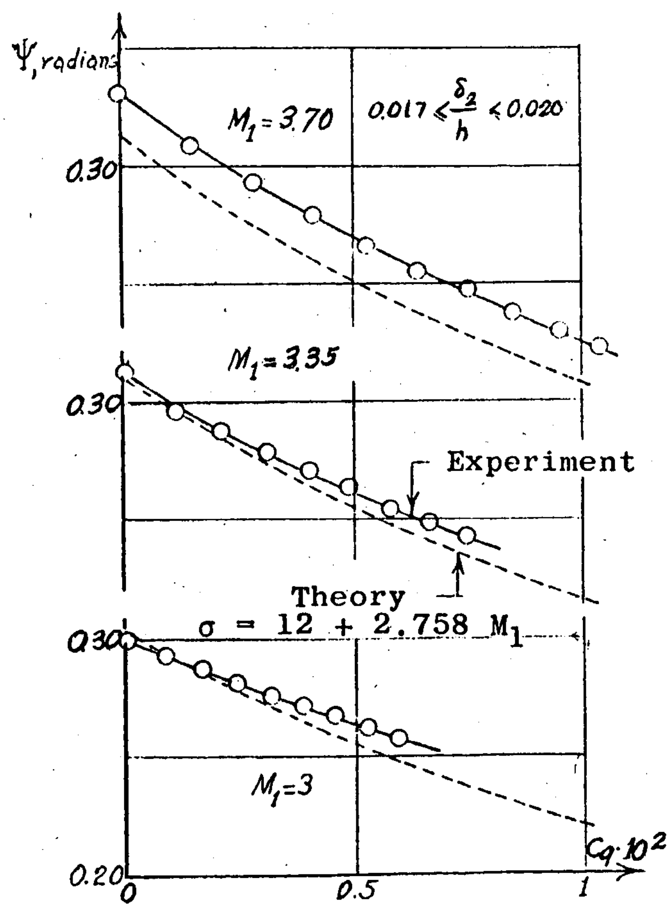


Figure 13

Effect of Blowing on Reattachment Angle - Comparison of Theory with Experiment for Mach Number Range $3.00 \leq M_1 \leq 3.70$

For purposes of comparison the theoretical data are plotted on the same diagrams to show the results of applying the method expounded in Section II.2.2, wherein the empirical constant λ_q has been assigned the value unity.

Upon closer scrutiny of these results it may be seen that the effect of blowing on the reattachment angle is quite well predicted, in general, by the theoretical calculation, even though the supposition is being made in this particular theory that the boundary layer is negligible. The experimental points do not fall off the theoretical curves by much, nor do they deviate hardly at all from the shifted theoretical curves which are obtained from the ones shown by making the values at the origin coincide and then applying this $\Delta\Psi$ incremental value to other points at appreciable values of C_q.

It is worth noting as a general principle that when the Mach number increases the effects on the reattachment angle are considerably accentuated. Furthermore, it may be observed that when the local slopes of the curves, $\frac{d\Psi}{dC_q}$, become significantly reduced, which is when C_q has grown large, then this condition constitutes a flow regime which is no longer legitimately covered by the linearization assumption, with consequent restriction in the universality of the application of this approximation.

It may be affirmed, likewise, from this comparison of theory and experiment that the initial slopes $\left(\frac{d\Psi}{dC_q}\right)_{C_q = 0}$ of the experimental curves are in general substantially smaller than those predicted by the theory. This deviation from the expected behavior may be attributed mainly to the presence of an appreciable boundary layer in the physical experiment, as will be more clearly brought out in subsequent discussion of this phase of the research on the reattachment problem.

II.4.3 - Effect of the Boundary Layer and Comparison with the Effect Produced by Secondary Injection

In the experimental set-up selected for test the two Mach numbers downstream and upstream of the corner expansion, M_1 and M_o, are practically the same, and they had the common value 3.0. Consequently, hardly any perturbation to the external inviscid flow was in operation to alter in any appreciable way the flow near the shoulder B of the rearward facing step (see Figure 11). Furthermore, under these conditions it is possible to determine the character of the boundary layer in the approaching flow by direct experimental probing of the velocity profiles.

Furthermore, in this arrangement where the pressures upstream and downstream of the shoulder B are kept equal, there is very little tendency for any disturbing side-wall boundary layer effects of any significance to intrude into the picture [see Reference 5]. Consequently, this elimination of disturbing factors gives these particular experiments a very desirable purity.

In Figure 14 are presented, first of all, the theoretical curve for the reattachment angle as effected by blowing, $\Psi\left(C_q = 0, \frac{\delta_2}{h} = 0\right)$ but without a boundary layer, and the corresponding graph showing the effect of the boundary layer, but without blowing, $\Psi\left(C_q = 0, \frac{\delta_2}{h}\right)$. In the first place it may be affirmed from the evident closeness of these curves that the effect on the reattachment angle is nearly the same whether produced by blowing or the boundary layer, provided the empirical shifting factors λ_δ and λ_q are equal. This close correspondence of the roles played by the blowing and by the boundary layer thus extends to a wider range of usefullness the rather remarkable property which was noted to hold true in mathematical rigor, in Section II.3.3, only for a small amount of blowing and for a thin boundary layer, i.e., for conditions only slightly removed from the basic reference flow configuration.

Consequently, it is now permissible to express the required angular deflection occurring at reattachment in the following manner

$$\Psi = f(C_q) + f\left(\frac{\delta_2}{h}\right) + f_i\left(C_q, \frac{\delta_2}{h}\right) \tag{1}$$

where the f function can serve to represent either the effect of blowing or the effect of the boundary layer, interchangeably, and where the f_i function is an adjustment term which indicates the interference effect of the two variables on the reattachment angle when operating simultaneously.

The above-mentioned behavior of the computed results may now be used to good advantage in explaining the experimental results which also have been plotted in Figure 14. The experimental data fall into three groups. There is a single piece of curve, $\Psi\left(\frac{\delta_2}{h}, C_q = 0\right)$, which represents the boundary layer effects for no blowing, while there are two groups of data of the form $\Psi\left(C_q, \frac{\delta_2}{h} = \text{const.}\right)$, which show the blowing effects at two different values for the boundary layer thickness (the open and closed circles in Figure 14).

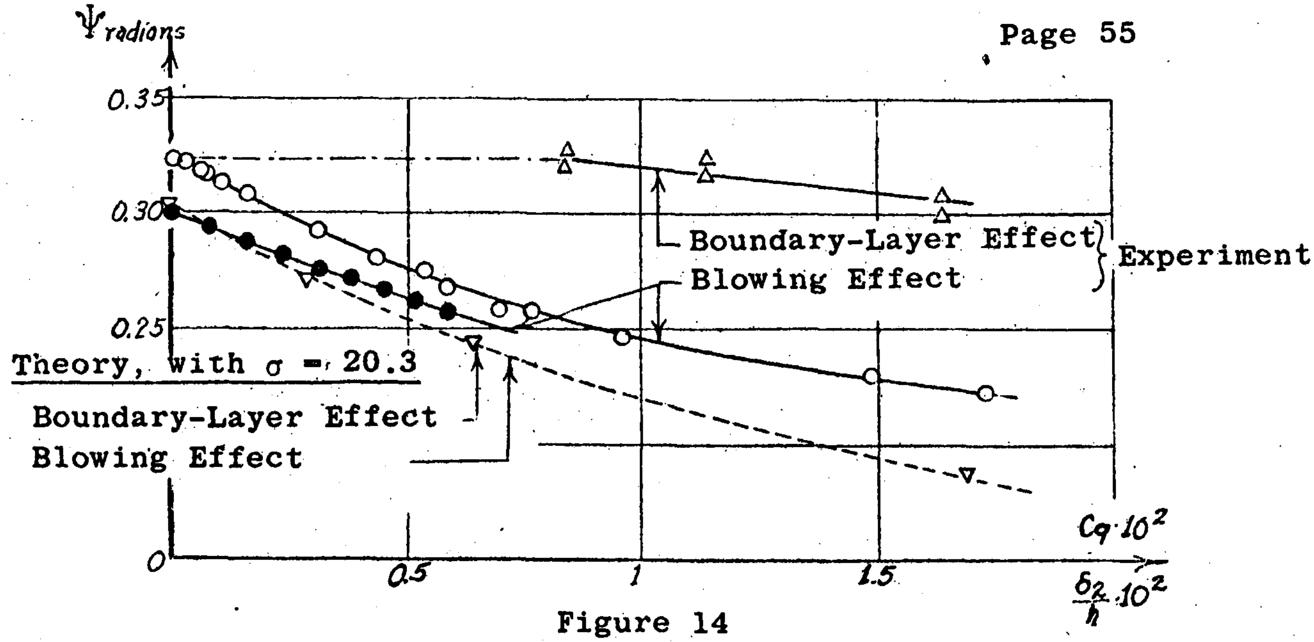


Figure 14
Effect of Blowing and of the Boundary Layer Size on Reattachment Angle - Comparison with Experiment at $M_1 = 3.0$

First of all the observation may be made that the effect of the secondary fluid injection on the reattachment angle $\Psi(C_q)$ is clear and unmistakable, particularly in the neighborhood of the origin; i.e., for flux parameters $C_q \leq 0.006$. When the theoretical curve (still being considered as a function of the secondary fluid injection parameter) is compared with the experimental data (open circles) it is obvious that the slope $\frac{d\Psi}{dC_q}$ observed from experiment is less than that predicted theoretically for values of the boundary layer thickness near $\frac{\delta_2}{h} \simeq 0.017$, while for thin effective boundary layers for which $\frac{\delta_2}{h} = 0.008$, the slopes of the theoretical and experimental curves turn out to be quite closely the same.

Such progressive shifts in the level of the experimental data may be interpreted as governed by the derivative $\frac{\partial}{\partial C_q}\left[f_i\left(C_q, \frac{\partial_2}{h}\right)\right]$ of the adjustment term, which accounts for the interference effect between the two variables, as represented in Eq. (1) of this section.

The practical usefulness of knowing the value of this derivative can, as well, be appreciated by noting its effect in causing the divergence between the experimental and theoretical results presented in Figures 12 and 13.

This information has valuable utility in establishing a working rule which will be valid within engineering accuracy. The function $f(C_q)$ which represents the effect of secondary fluid injection on the attachment angle evidently serves very well to stand for the real variation of Ψ with appreciable amounts of blowing, up to as intense a rate as that indicated by $C_q = 0.02$, even though a fairly sizeable value of the ratio of boundary layer to step height is allowed to come into play. This convenient representation of the true state of affairs would appear to imply that, in first approximation, the interference adjustment term $f_i\left(C_q, \frac{\delta_2}{h}\right)$ must be relatively insignificant in comparison with the main governing function f.

In addition, it is easy to see in Figure 14 that there is a shift, $\Delta\Psi$, in the ordinate intercepts, between the two curves representing the two different values of the boundary layer thickness ratio, δ_2/h, used in the experiments. This shift logically can be attributed to the corresponding shift in the attachment angle produced by influence of the boundary layer, according to a law also represented by $f\left(\frac{\delta_2}{h}\right)$, as indicated in Eq. 1 immediately above.

The experimental investigation of the boundary layer effect on the reattachment angle to determine $\Psi = f\left(\frac{\delta_2}{h}\right)$ was carried out for only a limited range of the variable $\frac{\delta_2}{h}$, and, unfortunately, the beginning of the interval was fairly far removed from the origin, because, with the particular experimental set-up available for these tests, it was not possible to reduce the δ_2/h ratio to any value smaller than 0.008.

Within the restrictions imposed by this narrow range of the variable that was investigated, it may be noted above all else that the sizeable displacement which is exhibited between the experimental and theoretical curves can, nevertheless, be accounted for by a simple vertical translation, represented by a correction factor $\Delta\Psi$. If such a translation were to be carried out on an analytic basis it would appear reasonable to try first of all to extrapolate the relatively short range of experimental data actually covered, in order to obtain an experimental value of the intercept, i.e., to obtain $\Psi\left(\frac{\delta_2}{h} = 0\right)$, where $\frac{\delta_2}{h} = 0$ is the ordinate axis. The theoretical function which was obtained for relating the deflection angle at reattachment to the boundary layer thickness ratio would now be brought into play in trying to produce an "adjusted" curve going through the so-determined experimental intercept and shifted this same constant amount for all points along the δ_2/h axis. Unfortunately, the intercept value for the experimental points found by such an extrapolation, $\overline{\Psi}$, is exceedingly far above the intercept value given by theory.

Although this result is only established on the basis of relatively few data it seems to be supported by additional evidence, inasmuch as the intercept values $\Psi\left(C_q = 0\right)$ which were found in Figures 12 and 13 for the case of large values of δ_2/h were just slightly above the theoretical values of Ψ for $M_1 > 2.35$.

It appears inescapable, therefore that the expression deduced as Eq. (8) in Section II.2.1 for the required angular deviation for reattachment should be reconsidered. With this end in mind it seems imperative to carry out new experimental studies also, in which much reduced values of the variable δ_2/h can be obtained. The experimental set-up should also be redesigned so as to eliminate, insofar as is possible, the three-dimensional disturbances originating from the boundary layers which grow up along the side walls.

III. Effect of Certain Curvature Factors on Reattachment

III.1 - Theoretical Approach to the Problem

III.1.1 - Essential Features of the Proposed Method

So far in the discussion the required angular deflection at reattachment has been found to have the following form

$$\Psi = \Psi\left(M_1,\ C_q,\ \frac{\delta_2}{h},\ w_m\right)$$

but the specific functional relationships which have been deduced have all hinged on the supposition that the flow after the corner expansion, represented by the Mach number M_1, is rigorously uniform, two-dimensional, and isobaric along the mixing region. It is the object of this part of the paper to extend the region of validity of this relationship in order to encompass those cases where certain curvature factors enter into the picture to alter the contour of the jet-like flow.

The method of characteristics is available for the study of the expansion processes taking place in the external inviscid flow as it passes the point B. This well-known method may be used to find the initial curvature X_1 of the jet boundary as a function of the several curvature factors which could come into play to upset the otherwise ordered flow downstream of the jet base. These relationships which apply only rigorously for a two-dimensional plane flow can continue to be used for all practical purposes when the configuration is actually a body of revolution. The initial curvature after the expansion, X_1, may be considered, thus, as the primary parameter representing the distortion of the jet-like flow from the fundamental reference configuration. If this initial curvature remains constant, then the angle of the flow, ψ, at any selected downstream position from B, having an abscissa denoted by x, will be proportional to this distance; i.e.,

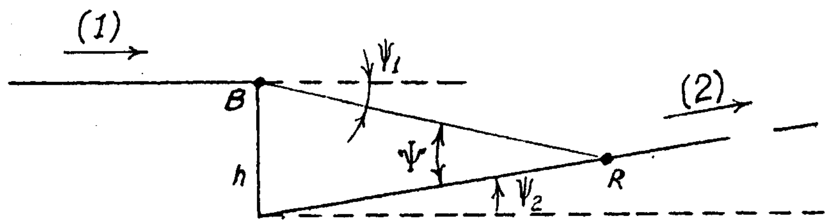
$$\psi = \psi_1 + x \cdot X_1.$$

In the more general case where the curvature X_1 varies as one progresses down the jet, the local flow angle may be expressed as

$$\psi = \psi_1 + \lambda \cdot x \cdot X_1$$

where λ will be a function of x in general, so that the degree of its deviation from unity will indicate the degree of variation of the local curvature from the nominal value X_1.

Let ψ_2 be the angle, measured with respect to the direction of the approaching undisturbed reference flow, that the reattached flow takes finally, and let h represent the height of the rearward facing step at detachment, so that the following sketch can serve to represent the elements of the flow configuration under study:



Although the effects of curvature factors on the reattachment may be complicated, it is best to proceed with an approximation analysis based on the a priori assumption that the behavior of the flow can be expressed in the form

$$(1) \qquad \boxed{\Psi(M_1,\ C_q\ \ldots) = \psi_2 - \psi_1 - \wedge \cdot \frac{h \cos \psi_2}{\sin \Psi} \cdot X_1}$$

where the quantity $\frac{h \cos \psi_2}{\sin \Psi}$ in this expression is approximately equal to the length of jet boundary from B to R. Note that X_1 is > 0 if the curvature of the BR boundary is concave upward.

It is being tacitly premised in writing Eq. (1) in the form given above that the perturbations produced in the flow are acting along the entire length of the BR boundary. If this is not so, then at least the x distance along which the curvature effect is noticeable will be less than BR, and this length should be introduced in place of the term h cos ψ_2/sin Ψ.

It will be demonstrated by supporting experimental results that a curvature influence function $\wedge(M_1)$ exists, such that Eq. (1) allows one to compute, to good approximation, what the reattachment conditions will be in a great variety of cases.

Before launching into a general frontal attack to solve this problem, let attention be directed first to the following simple particular experimental situation which will throw a good deal of light on the subject under examination (see Fig. 15).

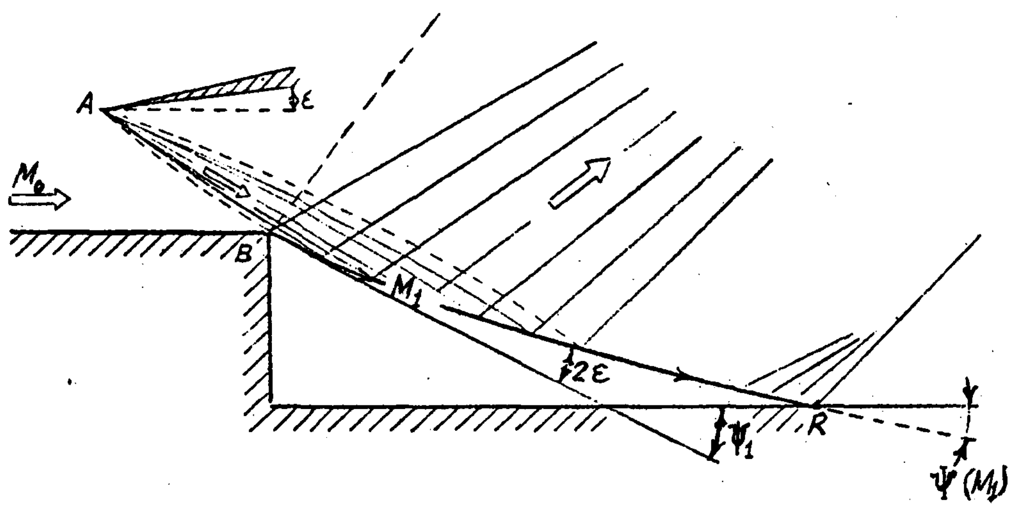


Figure 15

Diagram Showing How Curvature Affects Reattachment, in Two-Dimensional (Plane) Flow. Concept Is General, Despite Wedge Illustration.

Consider a two-dimensional wind-tunnel which produces a uniform flow at a Mach number of M_o upstream of the shoulder B. Furthermore, if nothing further intervenes, the flow downstream of B should follow the standard path of detachment and reattachment in such fashion as to represent the fundamental reference flow, where the presence here of any boundary layer is being ignored. If one now introduces into the flow, at a point such as A, a flat plate airfoil (with leading edge at A) which can be rotated to produce various negative incidence angles, ε, at will, then this plate at various settings will act to produce an expansion wave, downstream from B, which will play the role of a curvature factor in influencing the jet path contour.

The pertinent theoretical analysis for this situation is very elementary. If one considers first of all the case where the angular settings of the flat plate are only slightly inclined to the on-coming flow, then the entire expansion wave emanating from A will be reflected from the jet boundary, as shown in Figure 15. From the method of characteristics it will be seen at once that the angular induced deviation of the jet-like boundary is 2ϵ, if this boundary is considered to be an isobaric limiting line. Consequently, the angular direction at reattachment, Ψ, is given by the following expression (for the special case where ψ_2 has been taken to be zero):

$$\Psi(M_1) = P(M_o) - P(M_1) - 2\epsilon$$

On basis of this result it may be seen that M_1 is a function of ϵ, if the upstream undisturbed Mach number is assumed fixed in value.

As ϵ grows larger and larger (as the flat plate is rotated more and more from the zero position) the inviscid flow Mach number, M_1, increases, and thus the static pressure p_1 decreases. At the same time the stagnation point R moves upstream, while, conversely, the expansion fan spreads farther and farther downstream. Consequently, it turns out that a limiting incidence angle for the flat plate will be reached, after which no further change in the curvature effects should be noticeable. If it is imagined that the mixing zone along the jet-like boundary is very thin and if the region of reattachment is narrowed down to the point R, then it is clear that the first attainment of the limiting incidence angle mentioned above corresponds to the spreading of the last ray of the expansion fan from A just far enough downstream to first intersect the point R at this farthest downstream point of the jet-like boundary. The flow pattern for such an idealized situation can be determined by the method of characteristics.

An encouragingly close comparison between theory and experimental results is presented in Figure 16. The theoretical curve in this plot has been obtained under the simplifying hypothesis that the required angular deflection at reattachment, $\Psi(M_1)$, is not influenced at all by the curvature of the mixing zone.

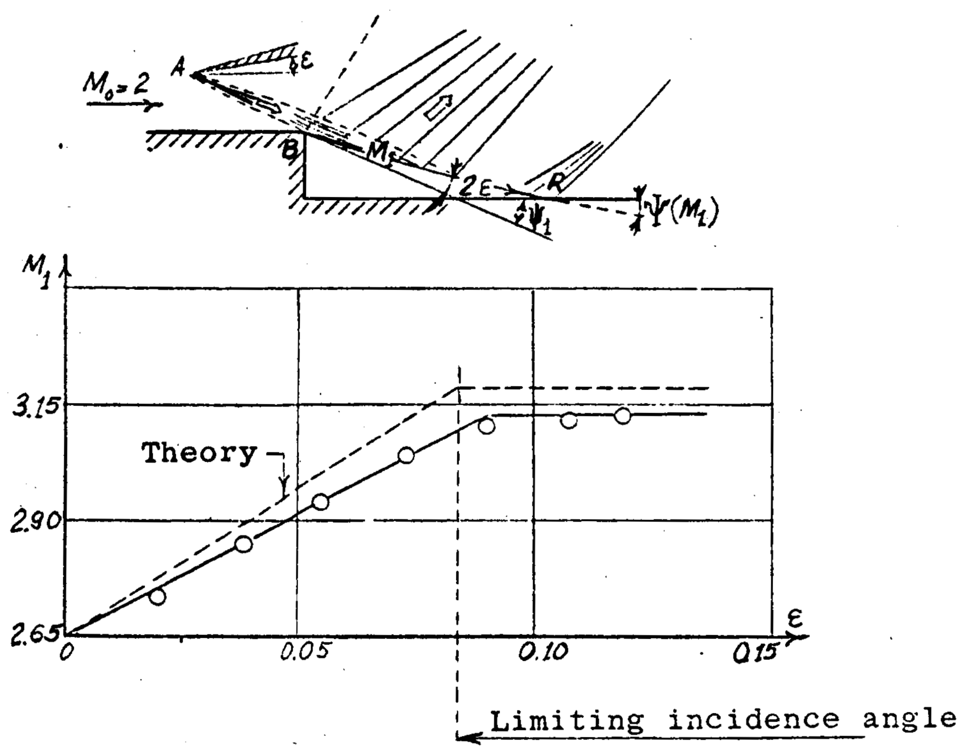


Figure 16

Effect of Induced Mach Number on Curvature of Jet Contour, for Two-Dimensional (Plane) Flow, at $M_o = 2.0$

It may be remarked that in this instance the total change brought about in the value of M_1, because of the expansion produced by the rotated flat plate, amounts to very sizeable increments. It would be reasonable to expect, therefore, that, if the curvature perturbation factor X_1 is of only moderate size, then the same method of calculation should certainly be satisfactory to account for such curvature disturbances brought about during the process of reattachment.

In the case of a flow surrounding a body of revolution, the rigorously exact working out of an equivalent procedure as that just described would entail a great deal of labor. On the other hand, it may be remarked that in the case of a body of revolution the intensity of the perturbations which might be encountered in any foreseeable practical case would tend to be much less strong than in the situation just described which pertained to two-dimensional flow.

These preliminary observations will suffice for the present to indicate the significance of Eq. (1). It is the object of the discussion to follow to retrace the development in more detail in order to make these relations more precise and useful.

III.1.2 - Determination of the Initial Curvature of the Streamlines in the Detached Jet-Like Flow

The physical situation to be analyzed first is that of a supersonic inviscid stream which is coursing along the forward surface of a rearward facing step, whether representing a two-dimensional configuration or a body of revolution with central axis coinciding with the coordinate axis $0\mathcal{X}$. In the region of the flow which lies upstream of the detachment occurring at the shoulder B the nature of the flow may be specified, in all respects having any bearing on the character of the subsequent local developments of interest in this study, by evaluation of the following parameters: Mach number, M_o; the direction, ψ_o, and the curvature X_o of the contour of the constraining wall at B; the tangential gradient of pressure, and the normal gradient of entropy there, denoted, respectively as

$$G_o = \frac{1}{p_o}\frac{\partial p_o}{\partial t} \quad \text{and} \quad s_{n_o} = \frac{1}{\gamma R}\left(\frac{\partial S}{\partial n}\right)_o .$$

These starting values can be computed by having recourse to rigorous calculational procedures such as given in Reference 3, or they could be determined experimentally by measurements made in the wind tunnel. In any event, it will be assumed for present purposes that these quantities have been evaluated in some fashion prior to the time when the need for their values crops up in what follows.

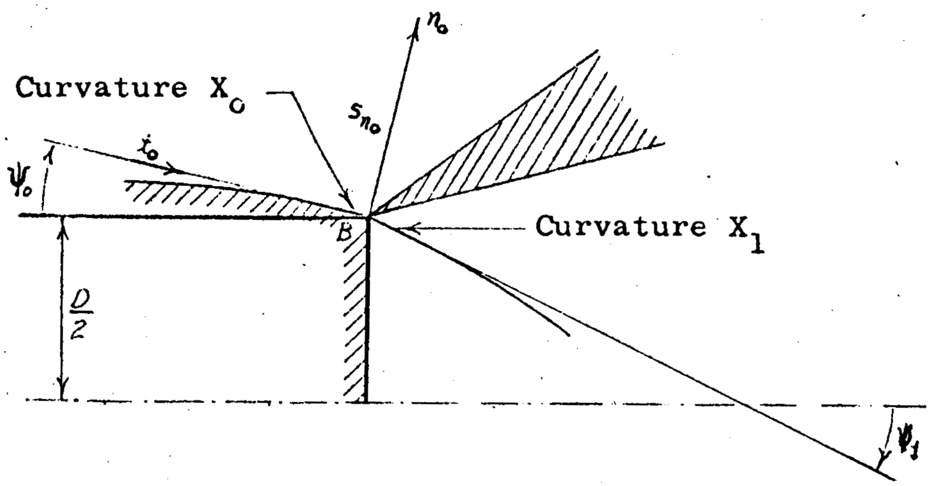


Figure 17

Geometric and Aerodynamic Factors Perturbing the Axially Symmetric Flow Past a Blunt-Ended Body

In order to describe how the flow behaves in expanding around the shoulder B (see Figure 17) some results previously discussed in Reference 3 will be recalled and, in addition, the following notation will be introduced:

$$\left\{\begin{array}{l} \alpha = \text{arc sin}\,\frac{1}{M} \\ d\mu - d\lambda = d\psi \\ d\mu + d\lambda = -\cot\alpha\,\frac{dV}{V} = dP \\ ds = \frac{dS}{\gamma R} \end{array}\right.$$

where λ and μ are the epicyclic coordinates pertaining to the hodograph plane, and where $s = S/(\gamma R)$ is the reduced entropy.

Now if ℓ and m represent the curvilinear coordinates of a point in the plane of the flow which is marked off by a grid of Mach lines, it may be shown that, for axially symmetric flows, the following system of governing equations is valid, where y represents the distance of the streamline from the axis of symmetry:

$$\left\{\begin{array}{l} \frac{\partial\lambda}{\partial\ell} = +\frac{\sin\alpha\cos\alpha}{2}\frac{\partial s}{\partial\ell} - \frac{\sin\alpha\sin\Psi}{2}\cdot\frac{1}{y} \\ \\ \frac{\partial\mu}{\partial m} = +\frac{\sin\alpha\cos\alpha}{2}\frac{\partial s}{\partial m} - \frac{\sin\alpha\sin\Psi}{2}\cdot\frac{1}{y} \\ \\ -\frac{\partial s}{\partial\ell} = \frac{\partial s}{\partial m} = s_n\sin\alpha. \end{array}\right.$$

In the case of a two-dimensional plane flow the same formulas govern the variables but the terms involving y merely must be dropped.

If X is now used to denote the local curvature of a streamline it is found that the following relation must hold true

$$2\ X \cos \alpha = \frac{\partial \mu}{\partial \ell} - \frac{\partial \lambda}{\partial m} + \sin^2 \alpha \cos \alpha\ s_n$$

and this is so regardless of whether the case in question refers to two-dimensional plane flow or to axially symmetric flow.

The theoretical problem which must now be solved consists then in carrying out an examination of the local flow phenomena which take place during an expansion around the shoulder B in order to be able to determine what the values of $\frac{\partial \mu}{\partial \ell}$, $\frac{\partial \lambda}{\partial m}$, and s_n become after this expansion occurs.

Such a study will be found to lead to the following formula for the initial curvature X_1 of the flow after the expansion has occurred:

$$(2) \qquad \boxed{X_1 = \alpha_1\ X_o + \frac{\alpha_2}{D} + a_3\ s_{n_o} + a_4\ G_o}$$

The coefficients, a_i, which have been introduced into this equation are described in detail in the Appendix*. It may be noted that they represent functions of the form

$$a_i = f_i(M_o,\ M_1,\ \psi_o).$$

In the case of two-dimensional plane flow the coefficient a_2 disappears, and the expression as given in the Appendix 2 for X_1 is rigorously true, unlike the case for axially symmetric flow, which is only an approximation.

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*The numerical evaluation of these coefficients has been carried out by the ONERA computing staff; the resulting tables may be obtained by inquiry to this office.

In the case of an axially symmetric flow the symbol D represents the diameter of the circular cross-section of the base from which the separation is taking place. The expression presented in Eq. (2) is not truly rigorous in this instance, because the initial curvature of the Mach lines at the shoulder B has been ignored in arriving at this formulation of the result.

In view of what is now known the problem is formally solved, because the governing parameters for the upstream flow are assumed known; i.e., it is taken for granted that M_o, ψ_o, s_{n_o}, G_o, and ψ_o are known; so that, by aid of Eq. 2, the curvature $X_1(M_1)$ becomes known. Of course, Eq. (1) may be used to find ψ_2, and, inasmuch as $\psi_1 = P(M_1) - P(M_o)$, then it follows that the complete solution of the problem may be worked out.

NOTE: If in a given flow there exists on the upstream side a prescribed expansion from M_o to M_1 and one assumes that the variations are going to be of weak intensity, indicated symbolically as δX_o, δs_{n_o}, and δG_o, then it may be deduced from Eq. (2) that the corresponding small change in the initial curvature of a jet streamline will be given in this case, as

$$\delta X_1 = a_1 \ \delta X_o + \frac{\delta a_2}{D} + a_3 \ \delta s_{n_o} + a_4 \ \delta G_o$$

where the result is valid to second order in X_o, s_{n_o}, and G_o.

In addition, the jet streamlines will not have any curvature. Furthermore, the coefficient a_2 will disappear in the two-dimensional plane case for which the upstream undisturbed flow is at Mach number M_o. By comparison, it may be observed that, for an axially symmetric flow with an undisturbed upstream Mach number of M_o ahead of the shoulder B, the curvature introduced by the body of revolution must be represented by the term $X_1 = a_2/D$.

III.1.3 - Importance of Curvature Effects on Subsequent Reattachment Phenomena

(a) Case of axially symmetric flow with uniform upstream Mach number of M_o.

The remark just made above furnishes the means for determining the initial curvature X_1 of the jet boundary as a function of the external Mach number attained after expansion, M_1. On the other hand, it is well-known that the angle ψ_1 that the jet boundary attains after the expansion is given by the Prandtl-Meyer relation for an expansion, which applies in this case as well as in case of two-dimensional plane flow. In consequence, one may compute ψ_1 from

$$\psi_1 = P(M_1) - P(M_o)$$

and, furthermore, Eq. (1) allows one to write that

$$\text{(3)} \qquad \Lambda_R = \frac{-\Psi\,(M_1,\ C_q\ \ldots) - P\,(M_1) + P\,(M_o)}{\dfrac{h\cos\psi_2}{\sin\Psi}\,X_1(M_1)}$$

provided the postulate of Section III.1.1 is adhered to.

Here the value of the Mach number in the external inviscid flow is assumed to be obtainable from experiment. It should be possible, for example, to determine the value for the empirical coefficient of curvature effect, $\Lambda_R(M_1)$, by measuring the base pressure on the body and then making use of Eq. (3).

By analysis of known experimental results, therefore, one will be able to establish the empirical constant in this function, and it will serve, consequently, as representing all flows which are axially symmetric and which do not deviate greatly from a uniform upstream flow.

(b) Case of two-dimensional plane flow which is not uniform.

The approach to this problem is made by first considering the reference case for which a uniform Mach number M_o holds upstream of the shoulder B, and for which the downstream external inviscid flow Mach number is M_1, and for which the required angular deflection at reattachment is $\Psi(M_1)$. To this reference flow with uniform upstream Mach number of M_o there is now applied a perturbation represented by X_o, s_{n_o}, and G_o, which is sufficiently extensive to influence the whole length of the jet boundary. As result of this perturbation there will be induced in the flow a curvature represented by X_1, and, according to Eq. 1, a change in Mach number, represented by $M_1 + \delta M_1$, will be introduced, which will obey the formula

$$\Psi_1' \cdot \delta M_1 = - \delta \psi_1 - \Lambda_p \frac{h \cos \psi_2}{\sin \Psi} X_1$$

where Ψ_1' is the partial derivative $\frac{\partial \Psi}{\partial M_1}$ of the required angular deflection for reattachment, as given in Figure 7, and where the increment $\delta \psi_1$ is given by the relation $\delta \psi_1 = P_1' \, \delta M_1$

in which $$P_1' = - \frac{1}{M_1} \frac{\sqrt{M_1^2 - 1}}{1 + \frac{\gamma - 1}{2} M_1^2} .$$

In this instance one then finds that the curvature coefficient is given by

$$\Lambda_p = - \frac{(P_1' + \Psi_1') \sin \Psi}{h \cos \Psi_2} \cdot \frac{\delta M_1}{X_1} .$$

This expression also affords the possibility of establishing the empirical value of Λ_p, the curvature coefficient due to non-uniform flow in the case of two-dimensional plane configurations. The empirical values of Λ_p can be obtained, for example, by use of the method described in Section II.1.1.

III.2 - Experimental Confirmations

III.2.1 - Procedure for Applying the Deductions now Elicited to the Case of Axially Symmetric Flows.

The curvature influence coefficient $\Lambda_R(M_1)$ has been determined by making use of a large compilation of experimental results obtained with cylindrical models having blunt bases. The pertinent Reynolds numbers were quite moderate, since $Re \leq 10^7$ [see Reference 6]. These results permitted the derivation of a law linking M_1 to M_o; i.e., $M_1 = f(M_o)$.

It is important to point out that this empirical law was developed by having recourse to experimental cases where the boundary layer, encountered just upstream of the bluntly cut-off base, was of only moderate thickness, although it was not entirely negligible nonetheless.

On the basis of these results it was then possible to calculate the initial curvature of the jet boundary by making use of Eq. (2) of Section III.1.2. Subsequently, then, the curvature influence coefficient $\Lambda_R(M_1)$ was obtained by use of Eq. (3), and the variation of this quantity, as a function of the Mach number, M_1, is displayed in Figure 18.

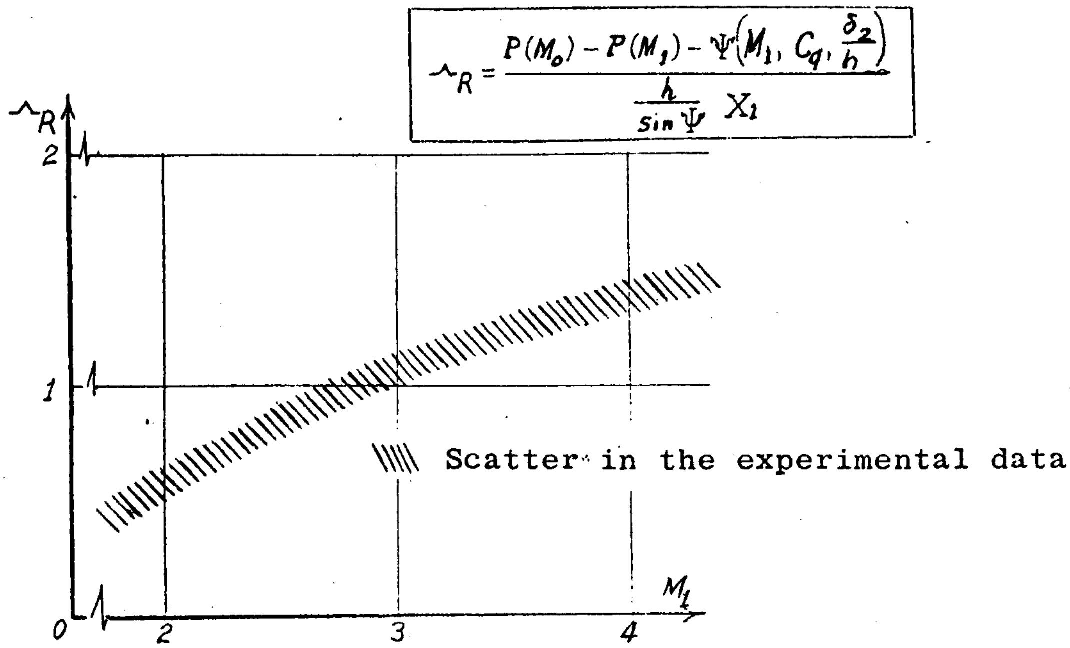


Figure 18

Curvature Influence Function, Λ_R, As Function of the External Flow Mach Number, M_1

This curvature influence coefficient may be considered to be a unique function of the Mach number, M_1, for flow configurations which do not differ much from the reference case (i.e., for a cylinder with a blunt base). Consequently, it is then possible to determine the effect of certain elementary perturbing influences on the reattachment phenomenon which arise downstream of an afterbody which has any shape whatsoever, so long as a linearization of the effects of the perturbations is justified. The perturbations in question may be pressure gradients or entropy gradients, for example, introduced by the non-cylindrical body shape.

The most immediate practical use to which this information may be put is the calculation of base pressures, that is to say, the calculation of the Mach number, M_1.

This particular calculation may be carried out by following either of the two acceptable methods given below. The essential assumption in either case is that the final direction of the downstream flow after reattachment must correspond to the same direction as the upstream approaching flow, for which $\psi_2 = 0$ in this instance.

(a) The most direct method of attack is to proceed according to the following algebraic steps

$$-\psi_1 - 2\Lambda_R \cdot h \cdot \sin\Psi \cdot X_1 = \Psi(M_1) \qquad \text{Given by Eq. (3)}$$

$$\psi_1 = \psi_o + P(M_1) - P(M_o) \qquad \text{Use of Prandtl-Meyer formula for expansion around corner B}$$

$$X_1(M_1) = a_1 X_o + \frac{a_2}{D_c} + a_3 s_{n_o} + a_4 G_o \qquad \text{Given by Eq. (2)}$$

by which means the solution for the Mach number M_1 can be most easily obtained by use of graphical aids.

(b) A variational technique may be resorted to which depends on finding deviations from a given experimental reference case. This method is also based on use of the same three formulas given above.

For the applications to be considered below, the experimental reference case is taken to be the one corresponding to a cylinder with a blunt base, for which it is true that $\psi_o = 0$.

III.2.2 - Numerical Examples

The effects to be investigated in these numerical applications are of two different categories. First, a calculation will be carried out to show the effect of boattailing (see Figures 19 to 21), for a Mach number range going from $M_o = 1.91$ to $M_o = 3.24$. Second, computations will be made to indicate what happens when there is a strong entropy gradient upstream of the base. This condition will be examined in the special case where the body is cylindrical with a well-rounded nose, the radius of curvature of which is quite large even at the tip ($r/R = 0.765$). The Mach number selected for this calculation is $M_\infty = 4$ (see Figure 22).

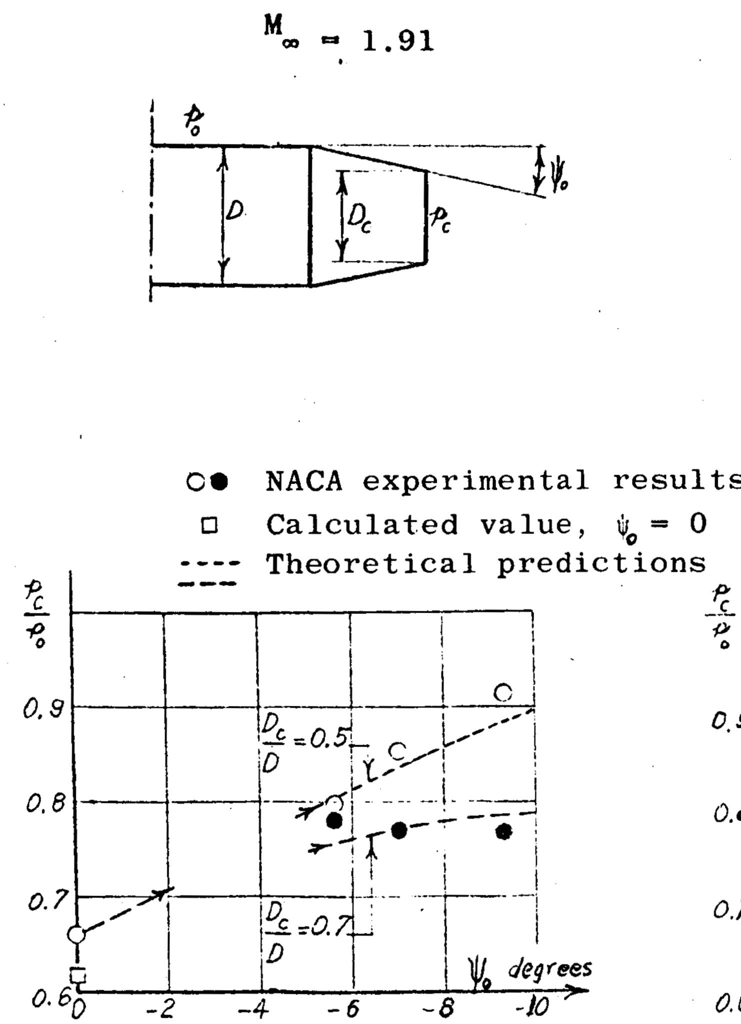


Figure 19
Axially Symmetric Flow - Effect of Boattailing

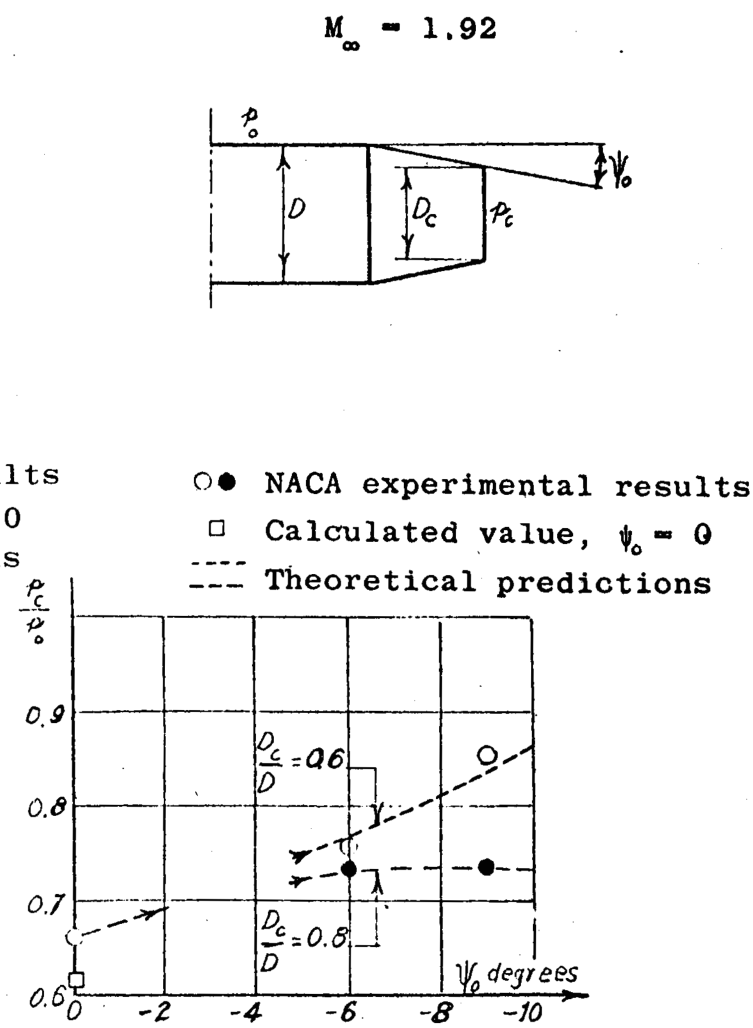
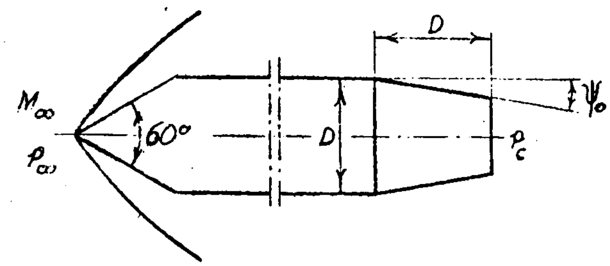


Figure 20
Axially Symmetric Flow - Effect of Boattailing



Mach Number, M_∞ = 3.24

○ N.O.L. experimental results

□ Calculated value, ψ_o = 0

---- Theoretical prediction

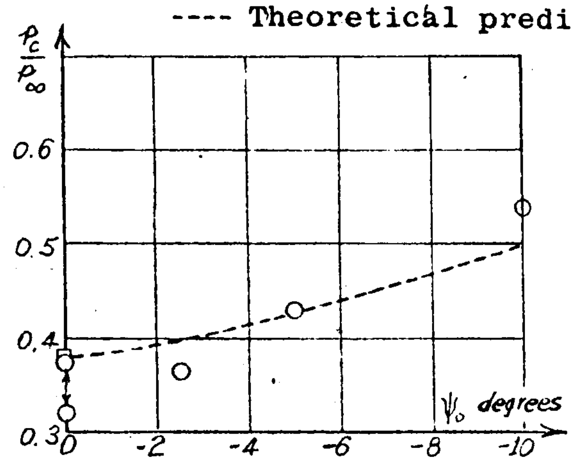


Figure 21

Axially Symmetric Flow - Effect of Boattailing

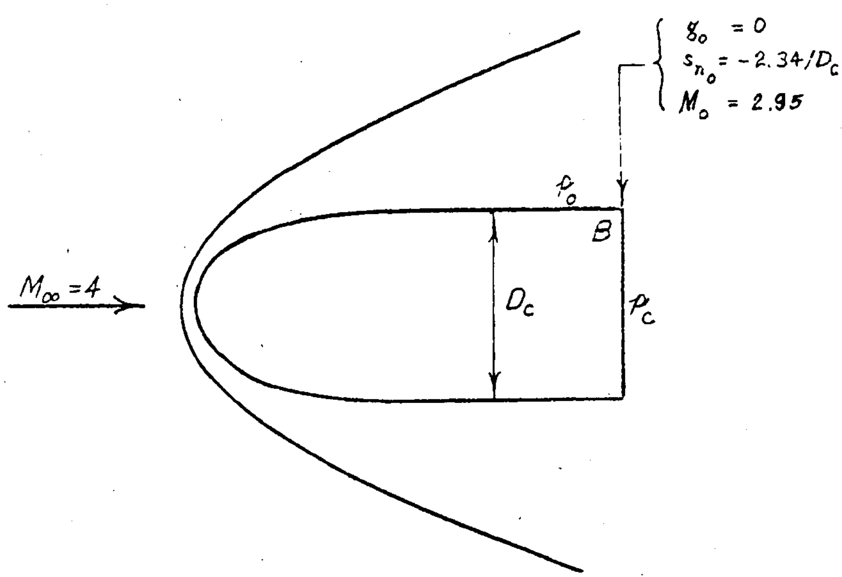


Figure 22

Axially Symmetric Flow - Effect of Entropy Gradients

		p_c/p_o	
s_{n_o}	M_o	Theoretical	Experimental
0	2.95	0.405	--
$-2.34/D_c$	2.95	0.37	0.38

The initial given conditions for these calculations (i.e., M_o, s_{n_o}, G_o) were determined in one of two ways, either by use of theory (i.e., by use of the method of characteristics, as expounded in Reference 3), or by measurement of the pressures on the afterbody to give G_o and M_o and by conducting a total pressure probing near the corner B in order to obtain s_{n_o}.

It may be fairly stated that the recommended method of calculation leads to satisfactory prediction of the base pressure, particularly when the variational procedure is followed (see Section III.2.1,b). This procedure has the advantage that it eliminates, for all intents and purpose, the effect of the boundary layer. Thus this case comes closer to being represented by the deduced equations because it was not possible to include the effect of the boundary layer in the derivation of the formula arrived at for describing the curvature influence coefficient Λ_R, as given in Section III.2.1.

In order to improve the accuracy of the direct approach it would be desirable to establish a law, through a systematic series of tests, which would give the effect of the boundary layer parameter δ_2/h upon the curvature influence coefficient, Λ_R.

From now on it appears that it would be justifiable, on basis of the results now obtained, to take it for granted in any application of this information to practical cases that

1^{st}: the required reattachment deflection angle $\Psi(M_1)$ is identical for axially symmetric flow and for two-dimensional plane flow

2^{nd}: there exists a curvature influence coefficient, $\Lambda_R(M_1)$, which is characteristic of axially symmetric flows.

IV. Conclusion

The preceding analysis has shown that the factors which influence the reattachment phenomena may be classified into two categories: On the one hand, there are the internal fundamental factors describing the local behavior of the flow which reflect the type and extent of the mixing that has occurred between the external flow and the internal trapped recirculatory flow. On the other hand, there are the external factors which act principally to change the geometric form (i.e., the curvature) of the jet-like flow issuing from off the shoulder of the body, so that in consequence the angle obtained by this flow, just before reattachment takes place, will be shifted from the value which would otherwise have arisen.

As result of the present studies having to do with the internal fundamental factors it has been possible to demonstrate the following significant theoretical results (this analysis was based primarily on the work presented by H. Korst in Reference 1, to which was added an additional hypothesis concerning the distribution of temperatures in the mixing zone):

(a) the situation in which the stagnation enthalpy in the mixing zone is variable from streamline to streamline may be handled by a simple device which converts this situation into one that is equivalent to the case where the enthalpy is constant throughout

(b) demonstration that the effects on reattachment of the presence of a boundary layer may be considered as equivalent to the proper amount of blowing into the dead-water region.

In the main, these theoretical deductions have been supported by confirming experimental data. At the same time, however, careful scrutiny of the results discloses that further research is still necessary in order to establish securely the value of the required angular deviation at reattachment in the limiting case where the thickness of the boundary layer in the approaching flow tends to vanish.

The external factors acting to vary the curvature of the jet boundary which were examined next in this investigation were: the effect produced by a pressure gradient or entropy gradient in the external flow, and, in the case of axially symmetric flows, the effect of the slope and curvature of the streamlines in the flow field which is produced just before separation from the base. A simple formula was obtained by resort to the method of characteristics which described adequately the effect, on the initial curvature of the jet-like flow off the base, produced by all the above-mentioned factors. This initial curvature of the jet-like flow can then be chosen as the sole parameter for representing the ultimate effect of these external factors on the reattachment angle.

If it is taken for granted that the required angular deviation at reattachment which was established in the case where there was no curvature along the jet boundary, for an isobaric pressure distribution, is still to remain valid in the envisioned cases, then the problem of determining the reattachment angle in these cases turns out to revert merely to finding a curvature influence coefficient. It is a simple matter to determine such a curvature influence coefficient by deducing it empirically from a limited number of pertinent and systematic wind-tunnel tests.

From the experimental results which were available it has been found possible to provide a preliminary approximation for such an influence coefficient in the case of axially symmetric flow. Experimental experience based on measurement of the base pressures of bodies of revolution has given very encouraging confirmation of the over-all effectiveness and correctness of this theory.

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Appendix I

Determination of η_1, η_j, $\left(\frac{d\eta_j}{d\eta_p}\right)_{\eta_p = 0}$

1. Basic Relations.

The theory of isobaric mixing is explained in Section II.1, at least in regard to the most essential features. It is merely necessary to recall here that this mixing theory leads to the velocity profile law given by the relation

$$\varphi = \frac{1}{2}\left[1 + \mathrm{erf}(\eta - \eta_p)\right] + \frac{1}{\sqrt{\pi}} \int_{\eta-\eta_p}^{\eta} \varphi_1\left(\frac{\eta - \beta}{\eta_p}\right) e^{-\beta^2}\, d\beta. \tag{1}$$

For present purposes it is helpful to start off with the additional relationship which is derived from the law governing the density profile across the boundary layer, which may be expressed as

$$\theta = \frac{\rho}{\rho_1} = \frac{T_1}{T} = \theta\ (u_1, w_m, \varphi).$$

2. Determination of η_1.

It is the purpose of this paragraph to determine the downward displacement, η_1, of the axis of the velocity profiles at each axial station x in such a way as to abide by the conservation of momentum requirement throughout the flow; i.e., for any value of the downstream location parameter η_p (see Section II.1.3). The conservation of momentum requirement was written near the beginning of Section II.1.3 of the main text. It is convenient to introduce the following set of non-dimensional variables at this point (see Figure 3):

$$\eta = \frac{y}{\delta} \cdot \eta_p = \zeta \eta_p$$

where $$\begin{cases} \eta_N = \dfrac{y_N}{\delta} \cdot \eta_p = \dfrac{y_N}{\delta} \cdot \eta_p = \zeta_N \cdot \eta_p \\ \\ \eta_N{}' = \dfrac{y_{N'}}{\delta} \cdot \eta_p{}' = \dfrac{y_N + y_1}{\delta} \cdot \eta_p = \eta_N + \eta_1 \end{cases}$$

Consequently, the statement of the conservation of momentum may then be converted into a relation that requires the equality of the two integrals:

$$\eta_p \int_0^{\zeta_N} \theta_1 \varphi_1^2 \, d\zeta = \int_{-\infty}^{\eta_N + \eta_1} \theta \varphi^2 \, d\eta$$

Inasmuch as $\theta_1 = 1$ and $\varphi_1 = 1$ for values of $\zeta > 1$, the integral on the left hand side may be written as

$$I_1 = \eta_N - \eta_p + \eta_p \int_0^1 \theta_1 \varphi_1^2 \, d\zeta .$$

In addition, it is always possible to choose η_N to be sufficiently large so that the values of $\theta - 1$ and $\varphi - 1$ will be less than an arbitrarily selected small value, ϵ, whenever $\eta \geq \eta_N$.

Hence, in this case the integral on the right hand side may be written as

$$I_2 = \eta_1 + \int_{-\infty}^{\eta_N} \theta \varphi^2 \, d\eta .$$

By invoking the equality of the two integrals it follows then that

$$\eta_1 = \eta_N - \eta_p + \eta_p \int_0^1 \theta_1 \, \varphi_1^2 \, d\zeta - \int_{-\infty}^{\eta_N} \theta \, \varphi^2 \, d\eta$$

or by rearranging the form

$$\eta_1 = - \eta_p \int_0^1 (1 - \theta_1 \, \varphi_1^2) \, d\zeta - \int_{-\infty}^{0} \theta \, \varphi^2 \, d\eta + \int_0^{+\infty} (1 - \theta \, \varphi^2) \, d\eta. \tag{2}$$

Now the last one of these integrals is convergent, so that by passing to the limit, $\eta_N \rightarrow \infty$, it turns out finally that

$$\eta_1 = - \eta_p \int_0^1 (1 - \theta_1 \, \varphi_1^2) \, d\zeta - \int_{-\infty}^{0} \theta \, \varphi^2 \, d\eta + \int_0^{\eta_N} (1 - \theta \, \varphi^2) \, d\eta.$$

3. Determination of η_j

3.1 Analytic Set-Up for the Analysis

The conservation of mass flow was discussed in Section II.1.4. When the non-dimensional form of the variables is introduced into the basic conservation equation and when the downward displacement of the profile axis as just described above is taken into account, then this continuity relationship may be written as

$$\eta_p \int_0^{\zeta_N} \theta_1 \, \varphi_1 \, d\zeta = \int_{\eta_j}^{\eta_N + \eta_1} \theta \, \varphi \, d\eta$$

and because $\Theta_1 = \varphi_1 = 1$ for $\zeta \geq 1$

while $\left.\begin{array}{l}\Theta - 1 < \varepsilon \\ \varphi - 1 < \varepsilon\end{array}\right\}$ for $\eta \geq \eta_N$

then the continuity equation may be rewritten as

$$\eta_p \int_0^1 \Theta_1\ \varphi_1\ d\zeta + \eta_N - \eta_p = \int_{\eta_j}^{\eta_N} \Theta\ \varphi\ d\eta + \eta_1 .$$

When the result just obtained in the preceding Section as Eq. (2) is introduced, it then follows that

$$\int_{\eta_j}^{\eta_N} \Theta\,\varphi\, d\eta = \eta_p \int_0^1 \Theta_1\ \varphi_1 (1 - \varphi_1)\ d\zeta + \int_{-\infty}^{\eta_N} \Theta\ \varphi^2\ d\eta$$

Upon adding and subtracting the term $\int_{-\infty}^{\eta_j} \Theta\ \varphi\ d\eta$ to both sides of this equation, it turns out that, upon passing to the limit $\eta_N \rightarrow \infty$,

$$\int_{-\infty}^{\eta_j} \Theta\ \varphi\ d\eta = -\ \eta_p \int_0^1 \Theta_1\ \varphi_1 (1 - \varphi_1) d\zeta + \int_{-\infty}^{+\infty} \Theta\ \varphi (1 - \varphi)\ d\eta \tag{3}$$

inasmuch as the integrals have now been made convergent.

If one cares to introduce the momentum thickness of the boundary layer defined in terms of $\varphi_1(\zeta)$, then the above equation may be reduced to the somewhat more neat form of

$$(3') \qquad \int_{-\infty}^{\eta_j} \theta \, \varphi \, d\eta = - \eta_p \frac{\delta_2}{\delta} + \int_{-\infty}^{+\infty} \theta \, \varphi \, (1 - \varphi) \, d\eta .$$

Once this relationship has been established it is not difficult to derive the expression given as Eq. (7) of Section II.1.4 in the main text.

3.2 Notes Concerning the Numerical Solution of Equation (3)

The following two integrals

$$\bar{I}_1 \, (\eta_j) = \int_{-\infty}^{\eta_j} \theta \, \varphi \, d\eta \cdot \frac{1}{1 - u_1^2}$$

and

$$\bar{I}_1(\infty) - \bar{I}_2(\infty) = \int_{-\infty}^{+\infty} \theta \, \varphi \, (1 - \varphi) \, d\eta \cdot \frac{1}{1 - u_1^2}$$

have already been tabulated by Korst for the arguments

$$\theta = \bar{\theta} = \frac{1 - u_1^2}{1 - u_1^2 \, \varphi^2} \quad \text{and} \quad \bar{\varphi} = \frac{1}{2} \, (1 + \operatorname{erf} \eta).$$

Consequently, the problem arises of using this information for obtaining the values of the integrals when the argument is $\varphi = \varphi \, (\eta, \eta_p)$, as given in Eq. (1) of the Appendix.

If η_p and $\varphi_1(\zeta)$ are given, then it is necessary, first of all, to compute the values of $\varphi(\eta)$ by means of Eq. (1).

Let $\eta(\varphi)$ be the corresponding, numerically evaluated, inverse function of $\varphi(\eta)$, and, furthermore, let $\overline{\eta}(\varphi)$ be the analytic inverse of $\varphi(\eta)$, and define a differential as

$$\delta\eta(\varphi) = \eta(\varphi) - \overline{\eta}(\varphi)$$

where φ is now taken to be the independent variable.

Upon making use of the shorthand notation that $K = \varphi(\eta_j)$, then Eq. (3) of this Appendix may be cast into the following form

$$\int_0^K \frac{\varphi \, d\varphi}{1 - u_1^2 \varphi^2} \left[\frac{d\overline{\eta}}{d\varphi} + \frac{d}{d\varphi} (\delta\eta) \right] = -\eta_p \frac{\delta_2}{\delta} \frac{1}{1 - u_1^2} + \int_0^1 \frac{\varphi(1-\varphi) \, d\varphi}{1 - u_1^2 \varphi^2} \left[\frac{d\overline{\eta}}{d\varphi} + \frac{d}{d\varphi} (\delta\eta) \right].$$

The expression just obtained may be given the following more simple symbolic representation

$$\overline{I}_1(K) + I_1'(K) = -\eta_p \frac{\delta_2}{\delta} \frac{1}{1 - u_1^2} + \overline{I}_1(\infty) - \overline{I}_2(\infty) + I_1'(\infty) - I'_2(\infty)$$

where the definitions have been introduced that

$$I_1' = \int_0^K \frac{\varphi \, d\varphi}{1 - u_1^2 \varphi^2} \frac{d}{d\varphi} (\delta\eta)$$

and $$I_1'(\infty) - I_2'(\infty) = \int_0^1 \frac{\varphi(1 - \varphi) \, d\varphi}{1 - u_1^2 \varphi^2} \frac{d}{d\varphi} (\delta\eta)$$

and where, by hypothesis, it has been stipulated that

$$\overline{I}_1(\overline{K}) = \overline{I}_1(\infty) - \overline{I}_2(\infty)$$

if $(\overline{K})$ represents the solution to Eq. (3) for the condition where $\eta_p = 0$.

Finally, therefore, because of these above-noted agreements, the equation which serves to determine K is the following:

$$\overline{I}_1(K) + \overline{I}_1'(K) = \overline{I}_1(\overline{K}) - \eta_p \frac{\delta_2}{\delta} + I_1'(\infty) - I_2'(\infty) \tag{3''}$$

where the quantities appearing on the right hand side are known.

This interpretation of the equations thus permits a relatively simple calculation to give the desired result provided one has recourse to the existing tabular data and after a numerical evaluation has been made for $\delta\eta(\varphi)$, I_1', and I_2'.

4. Determination of $\varphi' = \left(\frac{\partial\varphi}{\partial\eta_p}\right)_{\eta_p=0}$

(a) In the case that $\eta_p = 0$, then Eq. (1) reduces to

$$\overline{\varphi} = \frac{1}{2}\,(1 + \text{erf}\ \eta)$$

and Eq. (3) provides the corresponding value of η_j. Consequently, with this boundary coordinate determined, it follows that

$$\overline{K} = \overline{\varphi}(\overline{\eta}_j).$$

(b) In the case that $\eta_p \neq 0$, but where it still remains minuscule in size, one may rewrite Eq. (1) in the appropriate form of

(4) $$\varphi(\eta) = \overline{\varphi}(\eta) + \eta_p\, \varphi'.$$

On the other hand, since it is true that

$$\frac{\partial}{\partial \eta_p}\left[\mathrm{erf}\,(\eta - \eta_p)\right] = -\frac{2}{\sqrt{\pi}}\, e^{-(\eta - \eta_p)^2},$$

then for the condition of $\eta_p \longrightarrow 0$, this becomes

$$\frac{\partial}{\partial \eta_p}\left[\mathrm{erf}\,(\eta - \eta_p)\right] = -\frac{2}{\sqrt{\pi}}\, e^{-\eta^2}.$$

Now the integral appearing on the right hand side of Eq. (1) may be recast into the form

$$\frac{1}{\sqrt{\pi}}\int_{\eta-\eta_p}^{\eta} \varphi_1\left(\frac{\eta - \beta}{\eta_p}\right) e^{-\beta^2}\, d\beta = \frac{\eta_p}{\sqrt{\pi}}\int_0^1 \varphi_1(\zeta)\, e^{-(\zeta\,\eta_p - \eta)^2}\, d\zeta$$

provided the substitution is made that $\dfrac{\eta - \beta}{\eta_p} = \zeta$.

One may obtain without trouble the value of the derivative of this expression with respect to η_p, under the supposition that $\eta_p \longrightarrow 0$. The result is

$$\frac{e^{-\eta^2}}{\sqrt{\pi}}\int_0^1 \varphi_1(\zeta)\, d\zeta.$$

Consequently, it is found that

$$\varphi' = \left(\frac{\partial \varphi}{\partial \eta_p}\right)_{\eta_p = 0} = - \frac{e^{-\eta^2}}{\sqrt{\pi}} \left[1 - \int_0^1 \varphi_1(\zeta)\, d\zeta\right].$$

It is worth noting, in addition, that

$$\frac{e^{-\eta^2}}{\sqrt{\pi}} = \frac{d\bar{\varphi}}{d\eta}$$

and that $$1 - \int_0^1 \varphi_1\ (\zeta)\ d\zeta = \int_0^1 (1 - \varphi_1)\ d\zeta = \frac{\delta_{1_i}}{\delta}$$

where δ_{1_i} is the displacement thickness of the incompressible boundary layer having its velocity profile defined according to the law that $\frac{u}{u_1} = \varphi_1\left(\frac{y}{\delta}\right)$.

It turns out thus that the differential quotient of interest may be expressed as

$$\varphi' = - \frac{\delta_{1_i}}{\delta} \frac{d\bar{\varphi}}{d\eta} \ . \tag{5}$$

4.2 Determination of $\eta' = \left(\frac{d\eta_j}{d\eta_p}\right)_{\eta_p = 0}$.

When the derivative is taken of Eq. (3) of this Appendix, and if the results obtained above as Eq.(4) are made use of, it follows that

$$\frac{1}{1-u_1^2} \cdot \frac{\delta_2}{\delta} + \frac{K}{1-u_1^2 K^2} \cdot \frac{d\eta_j}{d\eta_p} = -\int_{-\infty}^{\eta_j} \frac{\varphi'(1+u_1^2 \bar{\varphi}^2)}{(1-u_1^2 \bar{\varphi}^2)^2} d\eta + \int_{-\infty}^{+\infty} \frac{\varphi'(1-2\bar{\varphi}+u_1^2 \bar{\varphi}^2)}{(1-u_1^2 \bar{\varphi}^2)^2} d\eta,$$

where the usual notation is employed defining $\bar{K}$ as $\bar{\varphi}(\eta_j)$.

If, now, one inserts into this expression the value of φ' obtained in Eq. (5), then the integrals appearing on the right hand side may be expressed in terms of $\bar{\varphi}$, used as the new valuable of integration. Hence

$$\frac{1}{1-u_1^2} \cdot \frac{\delta_2}{\delta} + \frac{K}{1-u_1^2 K^2} \cdot \eta' = \left[\int_0^1 \frac{2\bar{\varphi}\, d\bar{\varphi}}{(1-u_1^2 \bar{\varphi}^2)^2} - \int_K^1 \frac{1+u_1^2 \bar{\varphi}^2}{(1-u_1^2 \bar{\varphi}^2)^2} d\bar{\varphi}\right] \frac{\delta_{1_i}}{\delta}$$

This expression may be integrated directly without trouble, to give the final result, which turns out to be as follows, when all steps are completed:

$$\eta' = \frac{\delta_{1_i}}{\delta} - \frac{\delta_2}{\delta} \cdot \frac{1}{1-u_1^2} \cdot \frac{1-u_1^2 \bar{K}^2}{\bar{K}} . \tag{6}$$

Appendix 2

Initial Curvature of the Isobaric Jet Boundary
(Two-dimensional plane flow or axially symmetric flow)

1. Fundamental Relationships

The method of characteristics is based on the mathematical idea of being able to define the state of a supersonic flow at an arbitrary point Q by aid of three reduced variables λ, μ, s, which are linked to the direction of the flow ψ, to the Mach number M, and to the entropy S, by means of the relations

$$
(1) \qquad \begin{cases} \mu - \lambda = \psi + \text{constant} \\ \mu + \lambda = P(M) = \displaystyle\int - \frac{1}{M} \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma - 1}{2} M^2} \, dM \\ s = \dfrac{S}{\gamma R}. \end{cases}
$$

Let $\vec{\ell}$ and $\vec{m}$ designate unit vectors which are tangent to the Mach lines passing through Q, and let $\vec{t}$ and $\vec{n}$ represent unit vectors which are lying parallel and normal, respectively, to the velocity vector at the point Q. It is found, then, that the governing equations for the flow may be expressed as

$$
(2) \qquad \begin{cases} \dfrac{\partial \lambda}{\partial \ell} = \lambda_\ell = - \dfrac{\sin \alpha \sin \psi}{2y} + \dfrac{\sin 2 \alpha}{4} s_\ell \\ \dfrac{\partial \mu}{\partial m} = \mu_m = - \dfrac{\sin \alpha \sin \psi}{2y} + \dfrac{\sin 2 \alpha}{4} s_m \\ - s_\ell = + s_m = s_n \sin \alpha \end{cases}
$$

where y represents the radial distance of the point Q from the axis of revolution of the axially symmetric flow. In the event that the situation of interest is a two-dimensional plane flow, then the term involving y vanishes (it may be considered that the axis in the axially symmetric flow has retreated to an infinite distance).

It may be shown immediately that the curvature X of a streamline in such a flow, measured at the point designated by the number tripple (λ, μ, s), is given by the relation

$$2\ X \cos \alpha = \mu_{\ell} - \lambda_m + \sin^2 \alpha \cos \alpha\ s_n. \tag{3}$$

In a coordinate system based on the characteristic net (ℓ,m), the calculation of the streamline curvature X is easily carried out by aid of Eq. (3) provided the functions $\lambda(\ell,m)$, $\mu(\ell,m)$, $s(\ell,m)$ are specified.

Curvature of the Isobaric Jet Boundary

Along the isobaric boundary of the jet issuing off the rearward facing step under examination, for which it is true that M = constant, it is required that

$$\frac{\partial}{\partial t} (\lambda + \mu) = 0$$

where

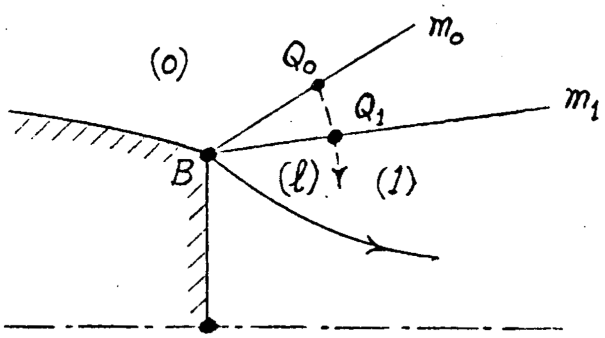
$$\frac{\partial}{\partial t} = \frac{1}{2 \cos \alpha}\left(\frac{\partial}{\partial m} + \frac{\partial}{\partial \ell}\right)$$

Consequently, if one makes use of Eqs. (1), (2), and (3) of this Appendix it follows that

$$2\ X \cos \alpha = \frac{\sin \alpha \sin \psi}{y} + \sin^2 \alpha \cos \alpha\ s_n - 2\ \lambda_m$$

2. Initial Curvature of the Isobaric Jet Right After the Corner Expansion

Let a zero subscript (o) denote upstream conditions, so that M_o, ψ_o, X_o, λ_o, μ_o, s_o represent the known condition of the flow ahead of the base region, before an expansion subsequently takes place at the corner B, for a flow configuration which is schematically represented by the accompanying sketch.



Also, let the differential coefficients, denoted by λ_{m_o}, μ_{m_o}, and s_{n_o} serve to define the known nature of any non-uniformity present in this upstream flow. Anywhere along the Mach line (m_o), which is indicated in the appended sketch as being coincident with the line BQ_o, one has that the following relations hold for an arbitrary point on m_o:

$$\begin{cases} \lambda(m_o) = \lambda_o + m_o \cdot \lambda_{m_o} = \lambda_o + \delta\lambda \\ \mu(m_o) = \mu_o + m_o \cdot \mu_{m_o} = \mu_o + \delta\mu \\ s(m_o) = s_o + m_o \sin\alpha_o \, s_{n_o} = s_o + \delta s. \end{cases}$$

Now let attention first be directed to what happens if an expansion denoted by (m_o, m_1) is allowed to take place around the corner B, but in an upstream flow field which is uniform, and denoted by (λ_o, μ_o). It is well known that under these circumstances (and even if the expansion is taking place around the base of a body of revolution) the expansion process obeys the Prandtl-Meyer law, within a region which is very close to the corner B. Very close means, in this case, that the Mach line λ is not supposed to be altered by the expansion, so that in this infinitesimally small area it is true that

$$\lambda_o = \lambda_1.$$

For such an expansion the local flow angle is given by the relation

$$\mu_1 = \mu_o + P(M_1) - P(M_o).$$

Finally, if the stipulation is obeyed that the expression is to take place isentropically, then the further condition may be premised that

$$s_1 = s_o$$

It is also convenient now to take it for granted that in a region which is infinitesimally close to the corner B the fan of successive Mach lines which issue from B will be indistinguishable, for all intents and purposes, from a sheaf of straight line rays. This hypothesis is rigorous for the case of an expansion taking place in a two-dimensional plane flow, but, of course, it will constitute a slight approximation in the case of axially symmetric flow.

With these understandings and definitions made, it is now possible to proceed to calculate the Mach line (1) which emanates from the point Q_o on m_o. The abscissa of the point Q_1 in the characteristic system is M_1, where Q_1 is located out along the ray m_1. This abscissa is given by

(5) $$\frac{m_1}{m_o} = \frac{H\ (\alpha_1)}{H\ (\alpha_o)}$$

$$\text{where} \quad \log H(\alpha) = -\ (\gamma + 1) \int \frac{\cot 2\ \alpha}{\gamma - \cos 2\ \alpha}\, d\alpha .$$

Meanwhile, the continuity equation for the mass flow allows one to obtain the areal dilation of the streamtubes as they undergo the expansion process. This area ratio may be evaluated from

(6) $$\frac{n_1}{n_o} = \frac{\Sigma(M_1)}{\Sigma(M_o)}$$

where $\Sigma(M)$ is used to represent the conventional mass flux ratios

$$\frac{A}{A_o} = \frac{\rho_\alpha\ a_c}{\rho u}.$$

With the foregoing suppositions in mind, it is now proposed to superimpose on the uniform upstream flow (M_o) just described a disturbance represented by the perturbations $\delta\lambda$, $\delta\mu$, and δs mentioned above. The disturbance under discussion is taken to act at the point Q_o and it is to be taken for granted that it is weak enough so that nothing is changed in regard to the net of Mach lines and flow lines, from what was true for the flow withour perturbations.

In consequence of these agreements, then, it will be found that, on the downstream side of the corner B, provided the relations given above in Eq. (2) are made use of, the following holds true:

$$\frac{s_{n_1}}{s_{n_o}} = \frac{\Sigma(M_o)}{\Sigma(M_1)} = \frac{\Sigma(\alpha_o)}{\Sigma(\alpha_1)}$$

and

$$\lambda(Q_1) - \lambda(Q_o) = m_1 \int_{\alpha_o}^{\alpha_1} \left[\frac{\sin\alpha \sin\psi}{2y} + \frac{\sin^2\alpha \cos\alpha}{2} s_{n_o} \frac{\Sigma(\alpha_o)}{\Sigma(\alpha)} \right] \times$$

$$\frac{\gamma + 1}{\gamma - \cos\alpha} \cdot \frac{H(\alpha)}{H(\alpha_1)} \cdot \frac{d\alpha}{\sin 2\alpha}.$$

By bringing in the relation given as Eq. (5) the above expression may be recast in slightly different form as

$$\lambda_{m_1} = \lambda_{m_o} \frac{H_o}{H_1} + \int_{\alpha_o}^{\alpha_1} \frac{\gamma + 1}{\gamma - \cos 2\alpha} \cdot \frac{H(\alpha)}{H(\alpha_1)} \cdot \frac{d\alpha}{\sin 2\alpha} \times$$

$$\left[\frac{\sin\alpha \sin\psi}{2y} + \frac{\sin^2\alpha \cos\alpha}{2} s_{n_o} \frac{\Sigma(M_o)}{\Sigma(\alpha)} \right].$$

In this expression the value of ψ is given by the Prandtl-Meyer expansion as

$$\psi = \psi_o + P(\alpha) - P(\alpha_o).$$

In the practical examples which are intended to be analyzed by use of this procedure it may be assumed that $\psi - \psi_o$ will be small enough so that it will be legitimate to make the approximation that

$$\sin \psi \cong \sin \psi_o + (P(\alpha) - P(\alpha_o)) \cos \psi_o$$

and the integral to be evaluated then may be broken down into terms of the following three auxiliary integrals

$$\begin{cases} A(\alpha) = \dfrac{\gamma + 1}{2} \displaystyle\int^{\alpha} \dfrac{H(\alpha)\, d\alpha}{\cos \alpha\, (\gamma - \cos 2\alpha)} \\ B(\alpha) = \dfrac{\gamma + 1}{2} \displaystyle\int^{\alpha} \dfrac{P(\alpha)\, H(\alpha)\, d\alpha}{\cos \alpha\, (\gamma - \cos 2\alpha)} \\ C(\alpha) = \dfrac{\gamma + 1}{2} \displaystyle\int^{\alpha} \dfrac{H(\alpha) \sin \alpha\, d\alpha}{\Sigma(\alpha)\, (\gamma - \cos 2\alpha)} \end{cases}$$

These integrals have been tabulated, so that the value of λ_{m_1} may be obtained without trouble, and then the initial curvature X_1 may be computed by aid of Eq. (4).

Before launching out on any such calculational procedure, however, it will be well worthwhile to realize that it is more convenient for making actual numerical applications to introduce some parameters which will be easier to work with than this value of λ_{m_o} for the Mach line upstream of the corner B about which the expansion is to take place.

When Eq. (3) is evaluated for the flow on the upstream side of the corner B it takes the form

$$2\ X_o \cos \alpha_o = \mu_{\ell_o} - \lambda_{m_o} + \sin^2 \alpha_o \cos \alpha_o\ s_{n_o}$$

and it is easy to verify that

$$2 \cos \alpha_o \frac{\partial}{\partial t} (\lambda + \mu) = \frac{2}{\gamma} \sin \alpha_o \cos^2 \alpha_o\ .\ G_o = \lambda_{m_o} + \mu_{\ell_o} - \frac{\sin \alpha_o \sin \psi_o}{y},$$

where $G_o = \frac{1}{p_o} \frac{\partial p_o}{\partial t}$ represents the tangential gradient of pressure at the point B but before encountering the detached flow zone.

These two expressions may then be used together to replace the quantity A_{m_o} by the more directly measurable quantities X_o and G_o. It will be found consequently that if $2\ y = D$, then the initial curvature of the jet boundary is given by the relation

$$\boxed{X_1 = a_1\ X_o + \frac{a_2}{D} + a_3\ s_{n_o} + a_4\ G_o}$$

where
$$a_1 = \frac{H_o \cos \alpha_o}{H_1 \cos \alpha_1}$$

$$a_2 = \frac{\sin \alpha_1 \sin \psi_1}{\cos \alpha_1} - \frac{H_o}{H_1} \frac{\sin \alpha_o \sin \psi_o}{\cos \alpha_1}$$

$$- \frac{1}{H_1 \cos \alpha_1} (\sin \psi_o - P_o \cos \psi_o)(A_1 - A_o)$$

$$- \frac{(B_1 - B_o) \cos \psi_o}{H_1 \cos \alpha_1}$$

$$a_3 = \frac{1}{2} \left[\sin^2 \alpha_1 \frac{\Sigma_o}{\Sigma_1} - \sin^2 \alpha_o \frac{\cos \alpha_o}{\cos \alpha_1} \cdot \frac{H_o}{H_1} - \frac{\Sigma_o}{H_1} \cdot \frac{C_1 - C_o}{\cos \alpha_1} \right]$$

and
$$- a_4 = \frac{1}{\gamma} \cdot \sin \alpha_o \cdot \frac{H_o}{H_1} \cdot \frac{\cos^2 \alpha_o}{\cos \alpha_1}.$$

<u>Note</u>:

In the case of a two-dimensional plane flow the term a_2 must be dropped, and one may show that the above-given formula holds rigorously.

In the case of axially symmetric flow the formula given above is only an approximation because it does not take into account the curvature of the Mach lines which is present in this case even though the basic flow is imagined to be unperturbed before the expansion.

Finally, in the case where the expansion taking place at the corner B occurs through means of the mirroring expansion fan, denoted by (ℓ_o, ℓ_1), then the foregoing formulas for the constants a_2 and a_4 need to be modified slightly, to read now:

$$a_2 = -\frac{\sin \alpha_1 \cdot \sin \psi_1}{\cos \alpha_1} + \frac{H_o}{H_1} \cdot \frac{\sin \alpha_o \cdot \sin \psi_o}{\cos \alpha_1}$$

$$+ \frac{\sin \psi_o + P_o \cos \psi_o}{H_1 \cos \alpha_1} \cdot (A_1 - A_o)$$

$$- \frac{\cos \psi_o}{H_1 \cos \alpha_1} \cdot (B_1 - B_o)$$

$$a_4 = \frac{1}{\gamma} \sin \alpha_o \frac{H_o}{H_1} \cdot \frac{\cos^2 \alpha_o}{\cos \alpha_1} .$$